Nonlinear integral equations in the Uimin-Sutherland model

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Uimin-Sutherland model

- Category of models based on $sl(r|s)$ Lie superalgebra:

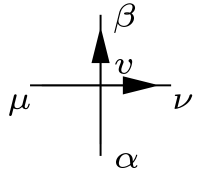
$$\mathcal{H}_0 = \sum_{i=1}^{L} \pi_{i,i+1} \quad ; \quad \mathcal{H} = \mathcal{H}_0 - \sum_{i=1}^{L} \sum_{\alpha=1}^{q} \mu_\alpha n_{i,\alpha}$$

with graded permutation operator:

$$\pi_{i,i+1} |\alpha_1 \dots \alpha_i \alpha_{i+1} \dots \alpha_L\rangle = (-1)^{\xi_{\alpha_i \alpha_{i+1}}} |\alpha_1 \dots \alpha_{i+1} \alpha_i \dots \alpha_L\rangle$$

- Classical counterpart is the rational limit of the *Perk-Schultz model*:

$$R^{\mu\nu}_{\alpha\beta}(v) := \delta_{\alpha\nu}\delta_{\mu\beta} + v \cdot (-1)^{\xi_{\alpha\mu}} \cdot \delta_{\alpha\beta}\delta_{\mu\nu} =$$



$$\mathcal{H}_0 = \frac{\mathrm{d}}{\mathrm{d}v} \ln \mathcal{T}(v) \Big|_{v=0} \quad \text{where} \quad \mathcal{T}^\beta_\alpha(v) := \sum_{\{\mu\}} \prod_{i=1}^{L} R^{\mu_i \mu_{i+1}}_{\alpha_i \beta_i}(v)$$

using the Yang-Baxter equation it follows that $[\mathcal{T}(v), \mathcal{H}_0] = 0$ for all $v \in \mathbb{C}$.

Uimin-Sutherland model

Some interesting models are of Uimin-Sutherland type:

- $sl(2)$ case:

  isotropic spin-$1/2$ Heisenberg chain, $\mathcal{H}_0 = \sum_{j=1}^{L}(2\mathbf{S}_j\mathbf{S}_{j+1} + \frac{1}{2})$
- $sl(3)$ case:

  spin-1 chain with biquadratic term, $\mathcal{H}_0 = \sum_{j=1}^{L}\left\{\mathbf{S}_j\mathbf{S}_{j+1} + (\mathbf{S}_j\mathbf{S}_{j+1})^2\right\}$
- $sl(4)$ case:

  supersymmetric point of $sl(2) \times sl(2)$ spin-orbital model,
  $\mathcal{H}_0 = \sum_{j=1}^{L}(2\mathbf{S}_j\mathbf{S}_{j+1} + \frac{1}{2})(2\boldsymbol{\tau}_j\boldsymbol{\tau}_{j+1} + \frac{1}{2})$
- $sl(2|1)$ case:

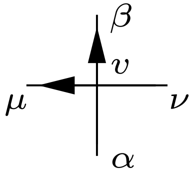
  supersymmetric t-J model (where $2t = J$),
  $\mathcal{H}_0 = -t\sum_{j,\sigma}\mathcal{P}(c_{j,\sigma}^{\dagger}c_{j+1,\sigma} + c_{j+1,\sigma}^{\dagger}c_{j,\sigma})\mathcal{P} + J\sum_{j}(\mathbf{S}_j\mathbf{S}_{j+1} - n_j n_{j+1}/4)$
- $sl(2|2)$ case: EKS model (introduced by Essler, Korepin & Schoutens, 92)

QTM approach to thermodynamics

Goal: Calculation of the partition function $Z = \mathrm{Tr}\,\mathrm{e}^{-\beta\mathcal{H}_0}$.

- Alternative transfer matrix with rotated vertex weights:

$$\overline{R}^{\mu\nu}_{\alpha\beta}(v) := R^{\alpha\beta}_{\nu\mu}(v) =$$



$$; \quad \overline{\mathcal{T}}^{\beta}_{\alpha}(v) := \sum_{\{\mu\}} \prod_{i=1}^{L} \overline{R}^{\mu_i \mu_{i+1}}_{\alpha_i \beta_i}(v)$$

- Connection to the Uimin-Sutherland model: $\mathcal{H}_0 = \frac{\mathrm{d}}{\mathrm{d}v} \ln \overline{\mathcal{T}}(v)\Big|_{v=0}$

- Taylor expansion at $v = 0$:

$$\ln \mathcal{T}(v) = \ln \mathcal{T}(0) + \mathcal{H}_0 \cdot v + \mathcal{O}(v^2)$$

$$\ln \overline{\mathcal{T}}(v) = \ln \overline{\mathcal{T}}(0) + \mathcal{H}_0 \cdot v + \mathcal{O}(v^2)$$

$$\Rightarrow \quad \mathcal{T}(v)\overline{\mathcal{T}}(v) = \underbrace{\mathcal{T}(0)\overline{\mathcal{T}}(0)}_{\text{shift operators}} \mathrm{e}^{2v\mathcal{H}_0 + \mathcal{O}(v^2)} = \mathrm{e}^{2v\mathcal{H}_0 + \mathcal{O}(v^2)}$$

QTM approach to thermodynamics

- Evaluate last equation at $u := -\beta/N$:

$$\mathcal{T}(u)\overline{\mathcal{T}}(u) = \mathrm{e}^{-\frac{2\beta}{N}\mathcal{H}_0 + \mathcal{O}((\beta/N)^2)}$$

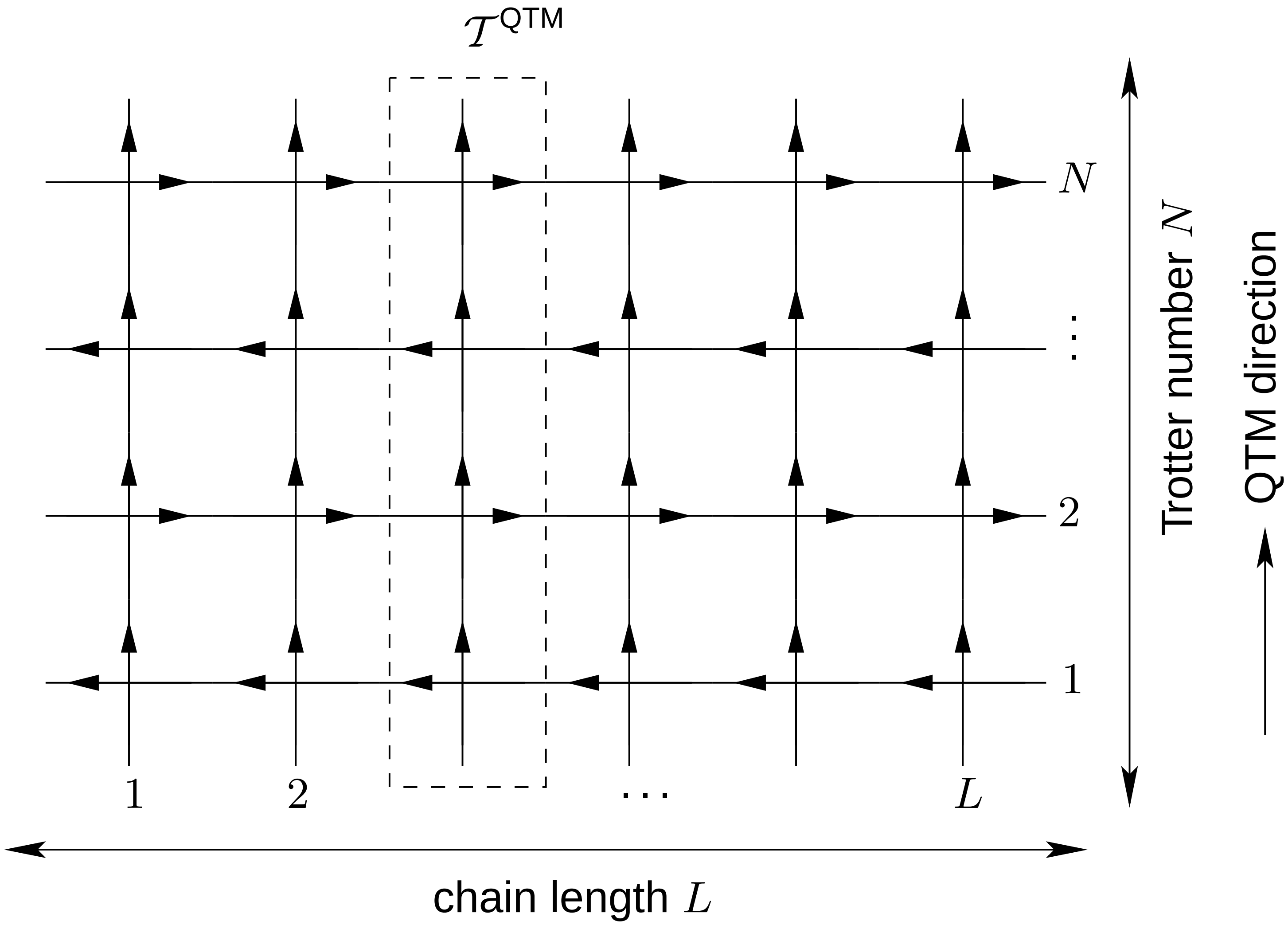
$$\Rightarrow \quad \lim_{N\to\infty} \mathrm{Tr}\left(\mathcal{T}(u)\overline{\mathcal{T}}(u)\right)^{N/2} = \lim_{N\to\infty} \mathrm{Tr}\left(\mathrm{e}^{-\frac{2\beta}{N}\mathcal{H}_0 + \mathcal{O}((\beta/N)^2)}\right)^{N/2}$$
$$= \mathrm{Tr}\,\mathrm{e}^{-\beta\mathcal{H}_0} = Z$$

- Partition function of the Uimin-Sutherland model is equal to the partition function of a $L \times N$ ”staggered” Perk-Schultz vertex model in the limit of N (Trotter number) to infinity:

$$Z = \mathrm{Tr}\,\mathrm{e}^{-\beta\mathcal{H}_0} = \lim_{N\to\infty} Z_{L,N} \quad ; \quad Z_{L,N} := \mathrm{Tr}\left(\mathcal{T}(u)\overline{\mathcal{T}}(u)\right)^{N/2}$$

- Problem: Z is still difficult to evaluate.

QTM approach to thermodynamics

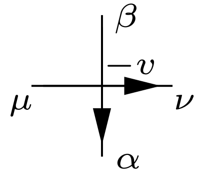


QTM approach to thermodynamics

Suitable definition of the *quantum transfermatrix (QTM)*:

$$\left(\mathcal{T}^{\mathsf{QTM}}\right)_{\alpha}^{\beta}(v) := \sum_{\{\mu\}} \prod_{j=1}^{N/2} R^{\mu_{2j-1}\mu_{2j}}_{\alpha_{2j-1}\beta_{2j-1}}(\mathrm{i}v+u)\widetilde{R}^{\mu_{2j}\mu_{2j+1}}_{\alpha_{2j}\beta_{2j}}(\mathrm{i}v-u)$$

$$\Rightarrow \quad Z_{L,N} = \mathrm{Tr}\left(\mathcal{T}(u)\overline{\mathcal{T}}(u)\right)^{N/2} = \mathrm{Tr}\left(\mathcal{T}^{\mathsf{QTM}}(0)\right)^{L}$$

- Rotated vertex weigths $\widetilde{R}^{\mu\nu}_{\alpha\beta}(v) := R^{\beta\alpha}_{\mu\nu}(-v) =$ 

- New spectral parameter v leads to commuting family of QTMs:

$$\left[\mathcal{T}^{\mathsf{QTM}}(v), \mathcal{T}^{\mathsf{QTM}}(v')\right] = 0 \quad \text{for all } v, v' \in \mathbb{C}$$

- QTM shares the $sl(r|s)$ symmetry of the Hamilton operator.

QTM approach to thermodynamics

- Gap between largest and next leading eigenvalue of the QTM
  $\Rightarrow$ Thermodynamics in the limit L to infinity given by largest eigenvalue:

$$f = -\lim_{L\to\infty} \frac{1}{L\beta} \ln Z = -\lim_{L\to\infty} \lim_{N\to\infty} \frac{1}{L\beta} \ln \mathrm{Tr} \left(\mathcal{T}^{\mathsf{QTM}}(0) \right)^L$$
$$= -\frac{1}{\beta} \lim_{N\to\infty} \ln \Lambda_{\mathsf{max}}(0)$$

- Next leading eigenvalues and eigenstates give correlation functions:

$$\langle \sigma_1 \sigma_{1+r} \rangle = \lim_{N\to\infty} \left\{ (\langle \Psi_0 | \sigma | \Psi_0 \rangle)^2 + \sum_{j=1}^{q^N-1} \langle \Psi_0 | \sigma | \Psi_j \rangle \langle \Psi_j | \sigma | \Psi_0 \rangle \cdot \mathrm{e}^{-r/\xi_j} \right\}$$

with correlation lengths:

$$\xi_j := \left\{ \ln \left(\frac{\Lambda_{\mathsf{max}}(0)}{\Lambda_j(0)} \right) \right\}^{-1}$$

QTM approach to thermodynamics

- Diagonalisation of QTM by Bethe ansatz leads to:

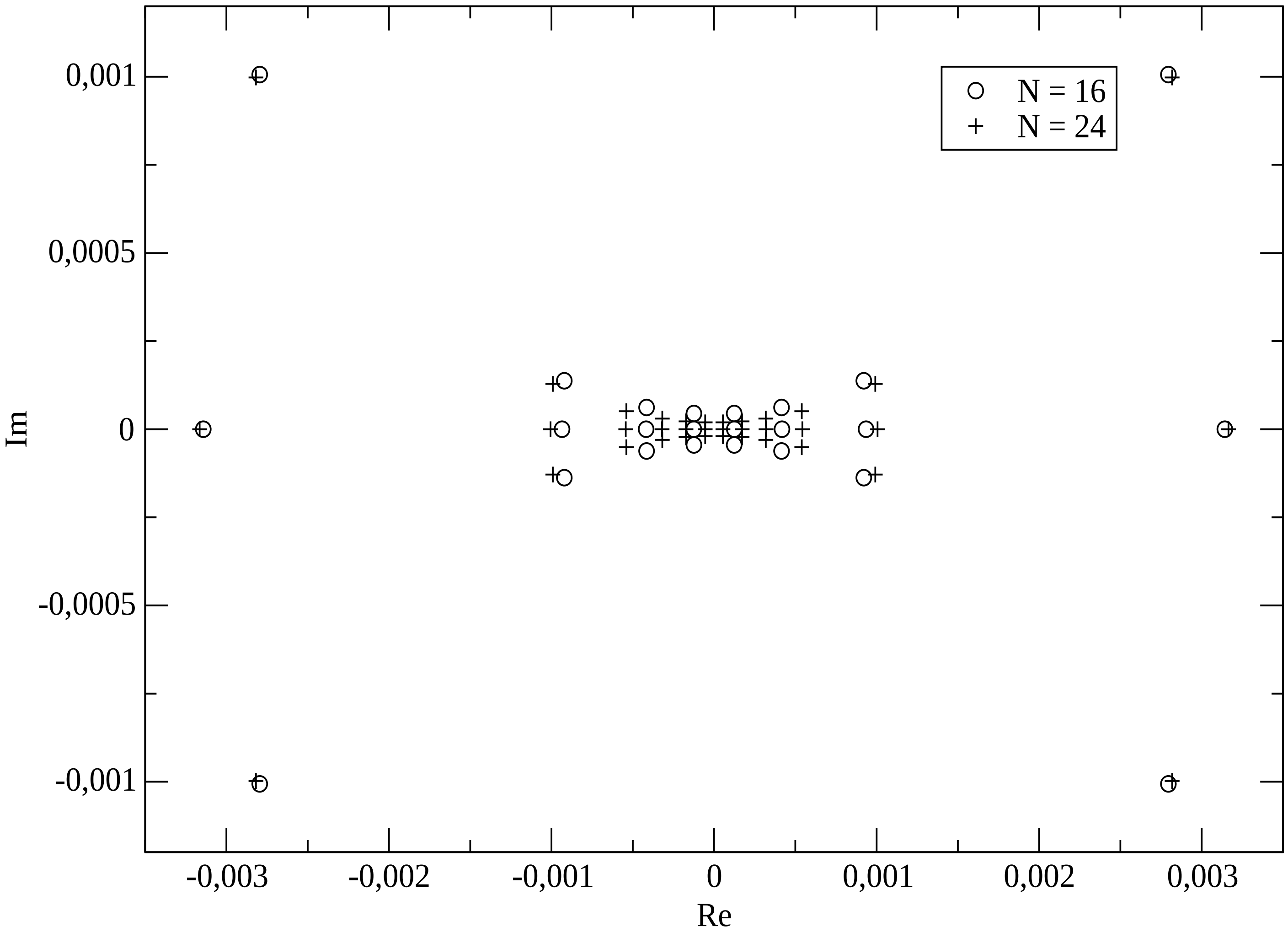
$$\Lambda(v) = \sum_{j=1}^{q} \lambda_j(v) \quad \text{where} \quad \begin{aligned} \lambda_1(v) &= \frac{q_1(v+\mathrm{i}\epsilon_1)}{q_1(v)} \phi_+(v)\phi_-(v-\mathrm{i}\epsilon_1)\mathrm{e}^{\beta\mu_1} \\ \lambda_j(v) &= \frac{q_{j-1}(v-\mathrm{i}\epsilon_j)}{q_{j-1}(v)} \frac{q_j(v+\mathrm{i}\epsilon_j)}{q_j(v)} \phi_+(v)\phi_-(v)\mathrm{e}^{\beta\mu_j} \\ \lambda_q(v) &= \frac{q_{q-1}(v-\mathrm{i}\epsilon_q)}{q_{q-1}(v)} \phi_+(v+\mathrm{i}\epsilon_q)\phi_-(v)\mathrm{e}^{\beta\mu_q} \end{aligned}$$

$$\left[\quad q_j(v) := \prod_{k_j=1}^{M_j} (v - v_{k_j}^j) \quad ; \quad \phi_\pm(v) := (v \pm \mathrm{i}u)^{N/2} \quad ; \quad q = r + s \quad \right]$$

- Pole free condition of the eigenvalue leads to *Bethe ansatz equations*:

$$\frac{\lambda_j(v_{k_j}^j)}{\lambda_{j+1}(v_{k_j}^j)} = -1 \quad \text{for all } 1 \le j \le q-1;\ 1 \le k_j \le M_j$$

Bethe-Ansatz roots for $sl(4)$ at $\beta = 0.01$



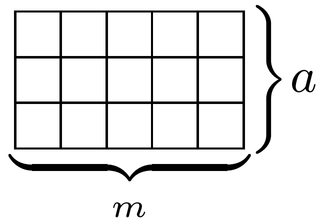
Motivation for the use of fusion hierarchy

- Bethe ansatz roots can not be treated by defining root densities in the limit N to infinity, as the distribution of roots remains discrete with a cluster point in the centre of the complex plane.
- A different approach is needed. Therefore one additionally defines a whole set of generalised transfer matrices, so called “fused transfer matrices”, where the representation, which builds the basis of the auxilliary space, is replaced by a higher dimensional representation of the same algebra.
- Fused transfer matrices still commute with each other by use of Yang-Baxter equation.
- Functional relations among fused transfer matrices help in the derivation of eigenvalues for the normal quantum transfer matrix.
- Relations have a deep mathematical origin, an exact sequence of Yangian modules (Kuniba, Nakanishi & Suzuki, 93).

Fusion rules for $sl(r|s)$ symmetric models

- Introduce transfer matrices for fused Boltzmann weights, where representation is characterised by a rectangular Young tableau:

$T_m^{(a)}(x)$ for the tableau



- Functional relation between transfer matrices of fusion type (T-system):

$$T_m^{(a)}\left(x-\tfrac{\mathrm{i}}{2}\right)T_m^{(a)}\left(x+\tfrac{\mathrm{i}}{2}\right)=T_m^{(a-1)}(x)T_m^{(a+1)}(x)+T_{m-1}^{(a)}(x)T_{m+1}^{(a)}(x)$$

$$\Rightarrow \quad \Lambda_m^{(a)}\left(x-\tfrac{\mathrm{i}}{2}\right)\Lambda_m^{(a)}\left(x+\tfrac{\mathrm{i}}{2}\right)=\Lambda_m^{(a-1)}(x)\Lambda_m^{(a+1)}(x)+\Lambda_{m-1}^{(a)}(x)\Lambda_{m+1}^{(a)}(x)$$

$$\Lambda_0^{(a)}(x)=\Lambda_m^{(0)}(x)=1 \quad ; \quad \Lambda_m^{(a)}(x)=0 \quad \text{for } a>r \text{ and } m>s$$

- The quantum transfer matrix is a special case, where $a=m=1$:

$$\mathcal{T}^{\mathrm{QTM}}(x)=T_1^{(1)}(x)$$

Fusion rules for $sl(r|s)$ symmetric models

- Compact notation, "Yangian analogue of Young tableaux":

$$\boxed{n} := \lambda_n(v) \quad \Rightarrow \quad \Lambda_1^{(1)}(x) = \sum_{n=1}^{r+s} \boxed{n}\Big|_{v=x}$$

- For general Young tableaux, we have:

$$\Lambda_m^{(a)}(x) = \sum_{\{n_{j,k}\}} \prod_{j=1}^{a} \prod_{k=1}^{m} \boxed{n_{j,k}}\Big|_{v=x+\mathrm{i}(j-a/2)-\mathrm{i}(k-m/2)}$$

$n_{j-1,k} \prec n_{j,k} \quad ; \quad n_{j,k-1} \preceq n_{j,k}$ if $n_{j,k}$ has positive grading

$n_{j-1,k} \preceq n_{j,k} \quad ; \quad n_{j,k-1} \prec n_{j,k}$ if $n_{j,k}$ has negative grading

- Example for $sl(3)$ case; $a = 2$, $m = 1$:

$$\Lambda_1^{(2)}(x) = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} + \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array} + \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array} = \lambda_1\left(x-\tfrac{i}{2}\right)\lambda_2\left(x+\tfrac{i}{2}\right)+\lambda_1\left(x-\tfrac{i}{2}\right)\lambda_3\left(x+\tfrac{i}{2}\right)+\lambda_2\left(x-\tfrac{i}{2}\right)\lambda_3\left(x+\tfrac{i}{2}\right)$$

Fusion rules for $sl(r|s)$ symmetric models

- Functional relations between eigenvalues lead to Y-system:

$$y_m^{(a)}(x) := \frac{\Lambda_{m-1}^{(a)}(x)\Lambda_{m+1}^{(a)}(x)}{\Lambda_m^{(a-1)}(x)\Lambda_m^{(a+1)}(x)}$$

$$Y_m^{(a)}(x) := \frac{\Lambda_m^{(a)}\left(x-\frac{\mathrm{i}}{2}\right)\Lambda_m^{(a)}\left(x+\frac{\mathrm{i}}{2}\right)}{\Lambda_m^{(a-1)}(x)\Lambda_m^{(a+1)}(x)} = y_m^{(a)}(x)+1$$

- With functional relations:

$$\frac{y_m^{(a)}\left(x-\frac{\mathrm{i}}{2}\right)y_m^{(a)}\left(x+\frac{\mathrm{i}}{2}\right)}{y_m^{(a-1)}(x)y_m^{(a+1)}(x)} = \frac{Y_{m-1}^{(a)}(x)Y_{m+1}^{(a)}(x)}{Y_m^{(a-1)}(x)Y_m^{(a+1)}(x)}$$

- In a concrete case, not all functions $y_m^{(a)}(x)$ can be defined and some are zero. Therefore the functional relations of the Y-system will differ slightly.
- By use of fourier transformation, one can obtain an infinite set of nonlinear integral equations from Y-system (equivalent to TBA equations).

$sl(2)$ case: exact truncation of the Y-system

- Functional relation for the eigenvalues:

$$\widetilde{\Lambda}_m^{(1)}\left(x-\tfrac{\mathrm{i}}{2}\right)\widetilde{\Lambda}_m^{(1)}\left(x+\tfrac{\mathrm{i}}{2}\right)=\widetilde{\Lambda}_m^{(0)}(x)\widetilde{\Lambda}_m^{(2)}(x)+\widetilde{\Lambda}_{m-1}^{(1)}(x)\widetilde{\Lambda}_{m+1}^{(1)}(x)$$

$$\widetilde{\Lambda}_m^{(0)}(x)\widetilde{\Lambda}_m^{(2)}(x)=\phi_-\left(x+\tfrac{m}{2}\mathrm{i}\right)\phi_+\left(x-\tfrac{m}{2}\mathrm{i}\right)\phi_-\left(x-\tfrac{m+2}{2}\mathrm{i}\right)\phi_+\left(x+\tfrac{m+2}{2}\mathrm{i}\right)$$

- This gives the Y-system:

$$y_m^{(1)}\left(x-\tfrac{\mathrm{i}}{2}\right)y_m^{(1)}\left(x+\tfrac{\mathrm{i}}{2}\right)=Y_{m-1}^{(1)}(x)Y_{m+1}^{(1)}(x)$$

- Exact truncation with suitable auxiliary functions at any level m:

$$Y_m^{(1)}(x)=B_m^{(1)}(x)\overline{B}_m^{(1)}(x)$$

$$B_m^{(1)}(x)=b_m^{(1)}(x)+1 \quad ; \quad \overline{B}_m^{(1)}(x)=\overline{b}_m^{(1)}(x)+1$$

$$b_m^{(1)}\left(x-\tfrac{\mathrm{i}}{2}\right)\overline{b}_m^{(1)}\left(x+\tfrac{\mathrm{i}}{2}\right)=Y_{m-1}^{(1)}(x)$$

$sl(2)$ case: exact truncation of the Y-system

- Auxiliary functions for $m = 1, 2, 3, 4$:

$$b_1^{(1)}(x) = \left.\frac{\boxed{1}}{\boxed{2}}\right|_{v=x+\frac{\mathrm{i}}{2}} \quad ; \quad \bar{b}_1^{(1)}(x) = \left.\frac{\boxed{2}}{\boxed{1}}\right|_{v=x-\frac{\mathrm{i}}{2}}$$

$$b_2^{(1)}(x) = \left.\frac{\boxed{1|1} + \boxed{1|2}}{\boxed{2|2}}\right|_{v=x+\frac{\mathrm{i}}{2}} \quad ; \quad \bar{b}_2^{(1)}(x) = \left.\frac{\boxed{1|2} + \boxed{2|2}}{\boxed{1|1}}\right|_{v=x-\frac{\mathrm{i}}{2}}$$

$$b_3^{(1)}(x) = \left.\frac{\boxed{1|1|1} + \boxed{1|1|2} + \boxed{1|2|2}}{\boxed{2|2|2}}\right|_{v=x+\frac{\mathrm{i}}{2}} \quad ;$$

$$\bar{b}_3^{(1)}(x) = \left.\frac{\boxed{1|1|2} + \boxed{1|2|2} + \boxed{2|2|2}}{\boxed{1|1|1}}\right|_{v=x-\frac{\mathrm{i}}{2}}$$

$$b_4^{(1)}(x) = \left.\frac{\boxed{1|1|1|1} + \boxed{1|1|1|2} + \boxed{1|1|2|2} + \boxed{1|2|2|2}}{\boxed{2|2|2|2}}\right|_{v=x+\frac{\mathrm{i}}{2}} \quad ; \quad \ldots$$

$sl(2)$ case: exact truncation of the Y-system

- Auxiliary functions factorise using the Bethe ansatz equations:

$$b_m^{(1)}(x) = \frac{q\left(x+\frac{m+2}{2}\mathrm{i}\right)\widetilde{\Lambda}^{(1)}_{m-1}(x)}{\phi_-\left(x+\frac{m}{2}\mathrm{i}\right)\phi_+\left(x+\frac{m+2}{2}\mathrm{i}\right)q\left(x-\frac{m}{2}\mathrm{i}\right)} \quad ;$$

$$\overline{b}_m^{(1)}(x) = \frac{q\left(x-\frac{m+2}{2}\mathrm{i}\right)\widetilde{\Lambda}^{(1)}_{m-1}(x)}{\phi_+\left(x-\frac{m}{2}\mathrm{i}\right)\phi_-\left(x-\frac{m+2}{2}\mathrm{i}\right)q\left(x+\frac{m}{2}\mathrm{i}\right)}$$

$$B_m^{(1)}(x) = \frac{q\left(x+\frac{m}{2}\mathrm{i}\right)\widetilde{\Lambda}^{(1)}_{m}\left(x+\frac{\mathrm{i}}{2}\right)}{\phi_-\left(x+\frac{m}{2}\mathrm{i}\right)\phi_+\left(x+\frac{m+2}{2}\mathrm{i}\right)q\left(x-\frac{m}{2}\mathrm{i}\right)} \quad ;$$

$$\overline{B}_m^{(1)}(x) = \frac{q\left(x-\frac{m}{2}\mathrm{i}\right)\widetilde{\Lambda}^{(1)}_{m}\left(x-\frac{\mathrm{i}}{2}\right)}{\phi_+\left(x-\frac{m}{2}\mathrm{i}\right)\phi_-\left(x-\frac{m+2}{2}\mathrm{i}\right)q\left(x+\frac{m}{2}\mathrm{i}\right)}$$

- Next step: Apply fourier transform to the logarithmic derivative of the auxiliary functions $y_1^{(1)},\ldots,y_{m-1}^{(1)},b_m^{(1)},\overline{b}_m^{(1)}$ and $Y_1^{(1)},\ldots,Y_{m-1}^{(1)},B_m^{(1)},\overline{B}_m^{(1)}$.
  $\Rightarrow$ Unknown functions $\widetilde{\Lambda}_1^{(1)},\ldots,\widetilde{\Lambda}_m^{(1)},q$ can be eliminated.

$sl(2)$ case: nonlinear integral equations

- Example: Truncation at level $m = 1$.
- Fourier transform of logarithmic derivative:

$$\widehat{f}(k) = \int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} \big[\ln f(x)\big] \mathrm{e}^{-\mathrm{i}kx} \, \frac{\mathrm{d}x}{2\pi}$$

- Results for $k < 0$:

$$\widehat{b}_1^{(1)}(k) = \mathrm{e}^{k/2} \widehat{\phi}_-(k) - \mathrm{e}^{k/2} \widehat{q}(k)$$
$$\widehat{\overline{b}}_1^{(1)}(k) = \mathrm{e}^{k/2} \widehat{\phi}_-(k) + \mathrm{e}^{3k/2} \widehat{q}(k) - \mathrm{e}^{3k/2} \widehat{\phi}_-(k) - \mathrm{e}^{k/2} \widehat{\phi}_+(k)$$
$$\widehat{B}_1^{(1)}(k) = \mathrm{e}^{-k/2} \widehat{\Lambda}_1^{(1)}(k) - \mathrm{e}^{k/2} \widehat{q}(k)$$
$$\widehat{\overline{B}}_1^{(1)}(k) = \mathrm{e}^{k/2} \widehat{\Lambda}_1^{(1)}(k) + \mathrm{e}^{k/2} \widehat{q}(k) - \mathrm{e}^{3k/2} \widehat{\phi}_-(k) - \mathrm{e}^{k/2} \widehat{\phi}_+(k)$$

(The case $k > 0$ can be treated in the same way.)

$sl(2)$ case: nonlinear integral equations

- Elimination of $\widehat{\Lambda}_1^{(1)}(k)$ and $\widehat{q}(k)$ gives:

$$\widehat{b}_1^{(1)}(k) = \frac{\mathrm{i}N\sinh(ku)}{2\cosh(k/2)} + \frac{\mathrm{e}^{-|k|/2}}{\mathrm{e}^{-k/2}+\mathrm{e}^{k/2}}\widehat{B}_1^{(1)}(k) - \frac{\mathrm{e}^{-k-|k|/2}}{\mathrm{e}^{-k/2}+\mathrm{e}^{k/2}}\widehat{\overline{B}}_1^{(1)}(k)$$

$$\widehat{\overline{b}}_1^{(1)}(k) = \frac{\mathrm{i}N\sinh(ku)}{2\cosh(k/2)} - \frac{\mathrm{e}^{k-|k|/2}}{\mathrm{e}^{-k/2}+\mathrm{e}^{k/2}}\widehat{B}_1^{(1)}(k) + \frac{\mathrm{e}^{-|k|/2}}{\mathrm{e}^{-k/2}+\mathrm{e}^{k/2}}\widehat{\overline{B}}_1^{(1)}(k)$$

- Taking the limit $N \to \infty$ and applying the inverse fourier transformation:

$$\ln b_1^{(1)}(x) = -\beta\left(V(x) + \frac{\mu_2-\mu_1}{2}\right) + K * \ln B_1^{(1)}(x) - K * \ln \overline{B}_1^{(1)}(x+\mathrm{i})$$

$$\ln \overline{b}_1^{(1)}(x) = -\beta\left(V(x) + \frac{\mu_1-\mu_2}{2}\right) - K * \ln B_1^{(1)}(x-\mathrm{i}) + K * \ln \overline{B}_1^{(1)}(x)$$

where $\quad V(x) = \dfrac{\pi}{\cosh(\pi x)} \quad ; \quad K(x) = \displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-|k|/2}}{\mathrm{e}^{-k/2}+\mathrm{e}^{k/2}}\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k$

$$\ln \Lambda_1^{(1)}(0) = -\beta\left(1 - 2\ln 2 - \frac{\mu_1+\mu_2}{2}\right) + V * \ln B_1^{(1)}(0) + V * \ln \overline{B}_1^{(1)}(0)$$

$sl(2)$ case: nonlinear integral equations

- General case (Suzuki, 99):

$$\begin{pmatrix} \ln y_1^{(1)}(x) \\ \ln y_2^{(1)}(x) \\ \vdots \\ \ln y_{m-1}^{(1)}(x) \\ \ln b_m^{(1)}(x) \\ \ln \overline{b}_m^{(1)}(x) \end{pmatrix} = \begin{pmatrix} -\beta V(x) \\ 0 \\ \vdots \\ 0 \\ -\beta \frac{\mu_2-\mu_1}{2} \\ -\beta \frac{\mu_1-\mu_2}{2} \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & V(x) & 0 & \cdots & 0 & 0 & 0 & 0 \\ V(x) & 0 & V(x) & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & V(x) & 0 & V(x) & V(x) \\ 0 & 0 & 0 & \cdots & 0 & V(x) & K(x) & -K(x+\mathrm{i}) \\ 0 & 0 & 0 & \cdots & 0 & V(x) & -K(x-\mathrm{i}) & K(x) \end{pmatrix} * \begin{pmatrix} \ln Y_1^{(1)}(x) \\ \ln Y_2^{(1)}(x) \\ \vdots \\ \ln Y_{m-1}^{(1)}(x) \\ \ln B_m^{(1)}(x) \\ \ln \overline{B}_m^{(1)}(x) \end{pmatrix}$$

Nonlinear integral equations in higher rank models

General strategy for higher rank models:

- Write down the functional relation of eigenvalues from the fusion rules (T-system).
- Derive the Y-system (from which TBA equations can be derived).
- Truncate the Y-system at a certain fusion level using suitable auxiliary functions.
- Derive a closed set of finitely many nonlinear integral equations by use of fourier transformation, eliminating all unknown functions.

Problem is, how to find suitable auxiliary functions:

- At the moment, some guessing is needed to get auxiliary functions leading to a closed set of nonlinear integral equations.
- Functional relations among auxiliary functions help in this process, but the exact origin of these relations is not well understood.

Higher rank models: $sl(3)$ case

- Functional relations from T-system:

$$\widetilde{\Lambda}_m^{(1)}\left(x-\tfrac{\mathrm{i}}{2}\right)\widetilde{\Lambda}_m^{(1)}\left(x+\tfrac{\mathrm{i}}{2}\right)=\widetilde{\Lambda}_{m-1}^{(1)}(x)\widetilde{\Lambda}_{m+1}^{(1)}(x)+\widetilde{\Lambda}_m^{(0)}(x)\widetilde{\Lambda}_m^{(2)}(x)$$
$$\widetilde{\Lambda}_m^{(2)}\left(x-\tfrac{\mathrm{i}}{2}\right)\widetilde{\Lambda}_m^{(2)}\left(x+\tfrac{\mathrm{i}}{2}\right)=\widetilde{\Lambda}_{m-1}^{(2)}(x)\widetilde{\Lambda}_{m+1}^{(2)}(x)+\widetilde{\Lambda}_m^{(1)}(x)\widetilde{\Lambda}_m^{(3)}(x)$$

- Rewritten as Y-system:

$$\frac{y_m^{(1)}\left(x-\tfrac{\mathrm{i}}{2}\right)y_m^{(1)}\left(x+\tfrac{\mathrm{i}}{2}\right)}{y_m^{(2)}(x)}=\frac{Y_{m-1}^{(1)}(x)Y_{m+1}^{(1)}(x)}{Y_m^{(2)}(x)}$$
$$\frac{y_m^{(2)}\left(x-\tfrac{\mathrm{i}}{2}\right)y_m^{(2)}\left(x+\tfrac{\mathrm{i}}{2}\right)}{y_m^{(1)}(x)}=\frac{Y_{m-1}^{(2)}(x)Y_{m+1}^{(2)}(x)}{Y_m^{(1)}(x)}$$

- Exact truncation at any level m:

$$Y_m^{(1)}(x)=B_m^{(1)}(x)\overline{B}_m^{(1)}(x)C_m^{(1)}(x)$$
$$Y_m^{(2)}(x)=B_m^{(2)}(x)\overline{B}_m^{(2)}(x)C_m^{(2)}(x)$$

Higher rank models: $sl(3)$ case

- Additional relations:

$$\frac{b_m^{(1)}\left(x-\frac{\mathrm{i}}{2}\right)\overline{b}_m^{(1)}\left(x+\frac{\mathrm{i}}{2}\right)}{c_m^{(2)}(x)}=\frac{Y_{m-1}^{(1)}(x)}{C_m^{(2)}(x)}$$

$$\frac{b_m^{(2)}\left(x-\frac{\mathrm{i}}{2}\right)\overline{b}_m^{(2)}\left(x+\frac{\mathrm{i}}{2}\right)}{c_m^{(1)}(x)}=\frac{Y_{m-1}^{(2)}(x)}{C_m^{(1)}(x)}$$

- Auxiliary functions for $m=1$ (Fujii & Klümper, 99):

$$b_1^{(1)}=\left.\frac{\boxed{1}}{\boxed{2}+\boxed{3}}\right|_{x+\mathrm{i}/2}\qquad \overline{b}_1^{(1)}=\left.\frac{\boxed{3}}{\boxed{1}+\boxed{2}}\right|_{x-\mathrm{i}/2}\qquad c_1^{(1)}=\left.\frac{\begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right)}\right|_{x}$$

$$b_1^{(2)}=\left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}}\right|_{x+\mathrm{i}/2}\qquad \overline{b}_1^{(2)}=\left.\frac{\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}}\right|_{x-\mathrm{i}/2}\qquad c_1^{(2)}=\left.\frac{\boxed{1}\,\boxed{3}}{\boxed{2}\left(\boxed{1}+\boxed{2}+\boxed{3}\right)}\right|_{x}$$

Higher rank models: $sl(3)$ case

- Generalisation is possible, analog to $sl(2)$ case:

$$b_2^{(1)} = \left.\frac{\boxed{1}\boxed{1} + \boxed{1}\boxed{2} + \boxed{1}\boxed{3}}{\boxed{2}\boxed{2} + \boxed{2}\boxed{3} + \boxed{3}\boxed{3}}\right|_{x+\mathrm{i}/2} \qquad \bar{b}_2^{(1)} = \left.\frac{\boxed{1}\boxed{3} + \boxed{2}\boxed{3} + \boxed{3}\boxed{3}}{\boxed{1}\boxed{1} + \boxed{1}\boxed{2} + \boxed{2}\boxed{2}}\right|_{x-\mathrm{i}/2} \qquad \ldots$$

- Auxiliary functions for truncation at any fusion level m:

$$b_m^{(1)}(x) = \frac{q_1\left(x + \frac{m+2}{2}\mathrm{i}\right)\widetilde{\Lambda}_{m-1}^{(1)}(x)}{\phi_-\left(x + \frac{m}{2}\mathrm{i}\right)X_{2,m}\left(x + \frac{\mathrm{i}}{2}\right)} \quad ; \quad \bar{b}_m^{(1)}(x) = \left(b_m^{(1)}(x)\right)^*$$

$$b_m^{(2)}(x) = \frac{q_2\left(x + \frac{m+3}{2}\mathrm{i}\right)\widetilde{\Lambda}_{m-1}^{(2)}(x)}{\phi_+\left(x + \frac{m+3}{2}\mathrm{i}\right)X_{1,m}(x)} \quad ; \quad \bar{b}_m^{(2)}(x) = \left(b_m^{(2)}(x)\right)^*$$

$$c_m^{(1)}(x) = \frac{q_1\left(x - \frac{m+2}{2}\mathrm{i}\right)q_2\left(x + \frac{m+2}{2}\mathrm{i}\right)\widetilde{\Lambda}_{m-1}^{(1)}(x)}{q_1\left(x + \frac{m}{2}\mathrm{i}\right)q_2\left(x - \frac{m}{2}\mathrm{i}\right)\widetilde{\Lambda}_m^{(2)}(x)}$$

$$c_m^{(2)}(x) = \frac{q_1\left(x + \frac{m+1}{2}\mathrm{i}\right)q_2\left(x - \frac{m+1}{2}\mathrm{i}\right)\widetilde{\Lambda}_{m-1}^{(2)}(x)}{q_1\left(x - \frac{m+1}{2}\mathrm{i}\right)q_2\left(x + \frac{m+1}{2}\mathrm{i}\right)\widetilde{\Lambda}_m^{(1)}(x)}$$

Higher rank models: $sl(3)$ case

- Same calculation as in $sl(2)$ case then yields:

$$\mathbf{y}(x) = -\beta \mathbf{d}(x) + \underline{\mathbf{K}} * \mathbf{Y}(x)$$

where:

$$\mathbf{y} = \left(\ln y_1^{(1)}, \ln y_2^{(1)}, \ln y_2^{(1)}, \dots, \ln y_{m-1}^{(2)}, \ln b_m^{(1)}, \dots, \ln c_m^{(2)}\right)^T$$

$$\mathbf{Y} = \left(\ln Y_1^{(1)}, \ln Y_2^{(1)}, \ln Y_2^{(1)}, \dots, \ln Y_{m-1}^{(2)}, \ln B_m^{(1)}, \dots, \ln C_m^{(2)}\right)^T$$

$$\mathbf{d}(x) = \left(V_1(x), V_2(x), 0, \dots, 0, a_b^{(1)}, a_{\bar{b}}^{(1)}, a_c^{(1)}, a_b^{(2)}, a_{\bar{b}}^{(2)}, a_c^{(2)}\right)^T$$

The kernel matrix $\underline{\mathbf{K}}$ is hermitian and has a regular block diagonal structure for the functions $\ln y_m^{(a)}(x)$ which is truncated in the lower right corner by a 6×6 block for the auxiliary functions $\ln b_m^{(1)}(x)$, ..., $\ln c_m^{(2)}(x)$. This last block of the matrix is furthermore made up of two different 3×3 blocks.

Higher rank models: $sl(4)$ case

- *Present work:* Auxiliary functions now also known for $sl(4)$ case.
- Derivation of nonlinear integral equations is completely analogous to the cases which have already been shown.
- It seems, that at least 14 functions are needed, which can be divided into two sets consisting of 4 functions and one set consisting of 6 functions.

$$Y_1^{(1)}(x) = B_1^{(1)}(x)\overline{B}_1^{(1)}(x)C_1^{(1)}(x)D_1^{(1)}(x)$$

$$Y_1^{(2)}(x) = B_1^{(2)}(x)\overline{B}_1^{(2)}(x)C_1^{(2)}(x)\overline{C}_1^{(2)}(x)D_1^{(2)}(x)E_1^{(1)}(x)$$

$$Y_1^{(3)}(x) = B_1^{(3)}(x)\overline{B}_1^{(3)}(x)C_1^{(3)}(x)D_1^{(3)}(x)$$

- *Conjecture:* Derivation can also be carried out for $sl(r)$. Number of auxiliary functions needed to truncate the Y-system is related to the dimension of the fundamental representations of the algebra.
  (e.g. $sl(2)$: 2; $sl(3)$: 3,3; $sl(4)$: 4,6,4; $sl(5)$: 5,10,10,5; . . .)

Higher rank models: $sl(2|1)$ and $sl(2|2)$ cases

- Program can also be executed for $sl(2|1)$ and $sl(2|2)$. Y-system is more complicated, but for both models the structure is similar to the $sl(2)$ case. This becomes obvious, if truncation is done at a fusion level greater than 1.
- Nevertheless fewer auxiliary functions are needed as for $sl(r)$ case of the same rank. For example one needs at least 3 functions for $sl(2|1)$, while 6 functions are needed for $sl(3)$.
- For the truncation at fusion level $m = 1$, the auxiliary functions for $sl(2|1)$ reduce to the known functions for the t-J-model (Jüttner & Klümper, 97):

$$b_1^{(1)} = \left.\frac{\boxed{1}}{\boxed{2}+\boxed{3}}\right|_{x+\mathrm{i}/2} \qquad \bar{b}_1^{(1)} = \left.\frac{\boxed{3}}{\boxed{1}+\boxed{2}}\right|_{x-\mathrm{i}/2} \qquad y_1^{(2)} = \left.\frac{\boxed{1}\,\boxed{3}}{\boxed{2}\left(\boxed{1}+\boxed{2}+\boxed{3}\right)}\right|_{x}$$

- Nonlinear integral equations for $sl(2|2)$ case (EKS model) are currently under investigation.

Conclusion and outlook

- Quantum transfer matrix approach to the Uimin-Sutherland model
- Fusion hierarchy leads to functional relations (T- and Y-system)
- Derivation of finite set of nonlinear integral equations for several models of Uimin-Sutherland type, now including $sl(4)$ and $sl(2|2)$ case.
  Generalisation for $sl(3)$ and $sl(2|1)$ to higher fusion levels

Open questions remain (present work):

- Generalisation for $sl(r)$ case: How to get the right auxiliary functions?
  Connection to the dimension of fundamental representations?
- Origin of additional functional relations between auxiliary functions.
  Is there a faster way to get the nonlinear integral equations?
- Numerical evaluation of NLIE for $sl(4)$ and $sl(2|2)$ case
- Connection to NLIE of Takahashi type, which have already been generalised to the $sl(r)$ case (Tsuboi, 03)