Nonlinear integral equations for the thermodynamics of the Uimin-Sutherland model

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1 Abstract

We investigate the thermodynamical properties of integrable one-dimensional spin-chains of Uimin-Sutherland type. This class includes several interesting models like the spin-$1/2$ Heisenberg model, the t-J model, the $SU(4)$-symmetric spin-orbital model and certain spin-ladder systems. We derive well-posed finite sets of nonlinear integral equations (NLIE) allowing for the numerical evaluation at arbitrary finite temperature. Analytical solutions are possible in the high- and low-temperature limits. In the low-temperature regime, we find divergences of the magnetic susceptibilities at critical fields and logarithmic singularities at zero magnetic field. In comparison to other recently derived NLIE, the evaluation at low temperature poses no problem in our formulation.

2 One-dimensional Uimin-Sutherland model

The Hamiltonian of the one-dimensional Uimin-Sutherland model is given by

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{ext}} = \sum_{j=1}^{L} \pi_{j,j+1} - \sum_{j=1}^{L}\sum_{\alpha=1}^{q} \mu_\alpha n_{j,\alpha}\,. \tag{1}$$

It acts on a one-dimensional lattice with L sites, where a q-state spin variable α_j is assigned to each site j. For each state α we have a grading $\varepsilon_\alpha = (-1)^{p(\alpha)} = \pm 1$. The local interaction operator $\pi_{j,j+1}$ permutes neighbouring spins on the lattice with respect to their grading,

$$\pi_{j,j+1}|\alpha_1\ldots\alpha_j\alpha_{j+1}\ldots\alpha_L\rangle = (-1)^{p(\alpha_j)p(\alpha_{j+1})}|\alpha_1\ldots\alpha_{j+1}\alpha_j\ldots\alpha_L\rangle\,, \tag{2}$$

where periodic boundary conditions are imposed. We have added external field terms $\mathcal{H}_{\text{ext}}$, where $n_{j,\alpha}$ counts the number of particles of type α sitting on site j, and μ_α is some general chemical potential. The model is known to be exactly solvable on the basis of the Yang-Baxter algebra. The classical counterpart is the rational limit of the two-dimensional Perk-Schultz model with Boltzmann weights

$$R^{\beta\nu}_{\alpha\mu}(v) = \delta_{\alpha\nu}\delta_{\mu\beta} + v\cdot(-1)^{p(\alpha)p(\mu)}\cdot\delta_{\alpha\beta}\delta_{\mu\nu}\,. \tag{3}$$

The model shows $sl(r|s)$ symmetry, where r and s are the total number of states with positive and negative grading ($q = r+s$), respectively.

In order to derive the thermodynamical properties of the Uimin-Sutherland model, we introduce the quantum transfer matrix (QTM),

$$\left(\mathcal{T}^{\text{QTM}}\right)^{\beta}_{\alpha}(v) = \sum_{\{\nu\}} \mathrm{e}^{\beta\mu_{\nu_1}} \prod_{j=1}^{N/2} R^{\beta_{2j-1}\nu_{2j}}_{\alpha_{2j-1}\nu_{2j-1}}(\mathrm{i}v+u)\,\widetilde{R}^{\beta_{2j}\nu_{2j+1}}_{\alpha_{2j}\nu_{2j}}(\mathrm{i}v-u)\,, \tag{4}$$

where N is the Trotter number, $u = -\beta/N$ and $\widetilde{R}^{\beta\nu}_{\alpha\mu}(v) = R^{\nu\alpha}_{\mu\beta}(-v)$. The partition function of the Uimin-Sutherland model can then be written in terms of the QTM,

$$Z = \operatorname{Tr}\mathrm{e}^{-\beta\mathcal{H}} = \lim_{N\to\infty}\operatorname{Tr}\left(\mathcal{T}^{\text{QTM}}(0)\right)^{L}. \tag{5}$$

This implies, that in the thermodynamic limit ($L\to\infty$) the free energy per site is solely given by the unique largest eigenvalue of the QTM at $v=0$ and $N\to\infty$,

$$f = -\lim_{L\to\infty}\frac{1}{L\beta}\ln Z = -\frac{1}{\beta}\ln\Lambda_{\max}(0)\,. \tag{6}$$

The QTM can be diagonalized by use of the Bethe ansatz (BA). This yields

$$\Lambda(v) = \sum_{j=1}^{q}\lambda_j(v) = \sum_{j=1}^{q}\phi_-(v)\phi_+(v)\frac{q_{j-1}(v-\mathrm{i}\varepsilon_j)}{q_{j-1}(v)}\frac{q_j(v+\mathrm{i}\varepsilon_j)}{q_j(v)}\mathrm{e}^{\beta\mu_j}\,, \tag{7}$$

where we have defined $\phi_\pm(v) = (v\pm\mathrm{i}u)^{N/2}$, $q_0(v) = \phi_-(v)$, $q_q(v) = \phi_+(v)$ and $q_j(v) = \prod_{k_j=1}^{M_j}(v - v^j_{k_j})$ for $j = 1,\ldots,q-1$. For each set j the $M_j \le N/2$ many complex BA roots $v^j_{k_j}$ have to fulfil the BA equations

$$\lambda_j(v^j_{k_j})/\lambda_{j+1}(v^j_{k_j}) = -1\,, \tag{8}$$

to ensure that all potential poles in the expression for $\Lambda(v)$ cancel. The actual numbers M_j depend on the eigenvalue of interest. To derive the BA roots corresponding to the largest eigenvalue of the QTM, we have to choose all $M_j = N/2$.

3 Derivation of the nonlinear integral equations

Equations (6) and (7) give the free energy of the model, but the task remains to solve the $(q-1)N/2$ many nonlinear equations (8) to get the right BA roots. This is quite cumbersome to do numerically and even impossible in the Trotter limit $N\to\infty$. Therefore, the goal is to encode the BA equations into a finite set of nonlinear integral equations (NLIE) for which this limit can be taken analytically.

The crucial point for the derivation of the NLIE is the definition of suitable auxiliary functions which involve sums and products of the terms $\lambda_j(v)$ appearing in the eigenvalue (7).

As an example, we treat the case of the $sl(4)$-symmetric Uimin-Sutherland model ($q=4$, all $\varepsilon_j = +1$) here, which for suitably chosen chemical potentials corresponds to the $SU(4)$ spin-orbital model,

$$\mathcal{H} = \sum_{j=1}^{L}(2\boldsymbol{S}_j\boldsymbol{S}_{j+1}+1/2)(2\boldsymbol{\tau}_j\boldsymbol{\tau}_{j+1}+1/2) - \sum_{j=1}^{L}\left(g_S h S^z_j + g_\tau h \tau^z_j\right)\,. \tag{9}$$

We need a set of 14 auxiliary functions for this case, which have been found very recently [1],

$$b^{(1)}_1(x) = \left.\frac{\boxed{1}}{\boxed{2}+\boxed{3}+\boxed{4}}\right|_{v=x+\mathrm{i}/2}\,,\qquad b^{(1)}_4(x) = \left.\frac{\boxed{4}}{\boxed{1}+\boxed{2}+\boxed{3}}\right|_{v=x-\mathrm{i}/2}\,,$$

$$b^{(1)}_2(x) = \left.\frac{\boxed{\substack{1\\2}}\cdot\left(\boxed{\substack{2\\3}}+\boxed{\substack{2\\4}}+\boxed{\substack{3\\4}}\right)}{\left(\boxed{\substack{1\\3}}+\boxed{\substack{1\\4}}\right)\cdot\left(\boxed{\substack{1\\2}}+\boxed{\substack{1\\3}}+\boxed{\substack{1\\4}}+\boxed{\substack{2\\3}}+\boxed{\substack{2\\4}}+\boxed{\substack{3\\4}}\right)}\right|_{v=x}\,,$$

$$b^{(1)}_3(x) = \left.\frac{\boxed{\substack{1\\3}}\cdot\boxed{\substack{3\\4}}}{\boxed{\substack{1\\4}}\cdot\left(\boxed{\substack{1\\3}}+\boxed{\substack{1\\4}}+\boxed{\substack{2\\3}}+\boxed{\substack{2\\4}}+\boxed{\substack{3\\4}}\right)}\right|_{v=x}\,,$$

$$b^{(2)}_1(x) = \left.\frac{\boxed{\substack{1\\2}}}{\boxed{\substack{1\\3}}+\boxed{\substack{1\\4}}+\boxed{\substack{2\\3}}+\boxed{\substack{2\\4}}+\boxed{\substack{3\\4}}}\right|_{v=x+\mathrm{i}/2}\,,$$

$$b^{(2)}_2(x) = \left.\frac{\boxed{\substack{1\\3}}\cdot\boxed{\substack{3\\4}}}{\left(\boxed{\substack{1\\4}}+\boxed{\substack{2\\4}}+\boxed{\substack{3\\4}}\right)\cdot\left(\boxed{\substack{2\\3}}+\boxed{\substack{2\\4}}+\boxed{\substack{3\\4}}\right)}\right|_{v=x+\mathrm{i}/2}\,,$$

$$b^{(2)}_3(x) = \left.\frac{\boxed{1}\cdot\boxed{4}}{\left(\boxed{2}+\boxed{3}\right)\cdot\left(\boxed{1}+\boxed{2}+\boxed{3}+\boxed{4}\right)}\right|_{v=x}\,,$$

$$b^{(2)}_4(x) = \left.\frac{\boxed{\substack{1\\2\\3}}\cdot\boxed{\substack{2\\3\\4}}}{\left(\boxed{\substack{1\\2\\4}}+\boxed{\substack{1\\3\\4}}\right)\cdot\left(\boxed{\substack{1\\2\\3}}+\boxed{\substack{1\\2\\4}}+\boxed{\substack{1\\3\\4}}+\boxed{\substack{2\\3\\4}}\right)}\right|_{v=x}\,,$$

$$b^{(2)}_5(x) = \left.\frac{\boxed{\substack{1\\2}}\cdot\boxed{\substack{2\\4}}}{\left(\boxed{\substack{1\\2}}+\boxed{\substack{1\\3}}+\boxed{\substack{1\\4}}\right)\cdot\left(\boxed{\substack{1\\2}}+\boxed{\substack{1\\3}}+\boxed{\substack{2\\3}}\right)}\right|_{v=x-\mathrm{i}/2}\,,$$

$$b^{(2)}_6(x) = \left.\frac{\boxed{\substack{3\\4}}}{\boxed{\substack{1\\2}}+\boxed{\substack{1\\3}}+\boxed{\substack{1\\4}}+\boxed{\substack{2\\3}}+\boxed{\substack{2\\4}}}\right|_{v=x-\mathrm{i}/2}\,,$$

$$b^{(3)}_1(x) = \left.\frac{\boxed{\substack{1\\2\\3}}}{\boxed{\substack{1\\2\\4}}+\boxed{\substack{1\\3\\4}}+\boxed{\substack{2\\3\\4}}}\right|_{v=x+\mathrm{i}/2}\,,\qquad b^{(3)}_4(x) = \left.\frac{\boxed{\substack{2\\3\\4}}}{\boxed{\substack{1\\2\\3}}+\boxed{\substack{1\\2\\4}}+\boxed{\substack{1\\3\\4}}}\right|_{v=x-\mathrm{i}/2}\,,$$

$$b^{(3)}_2(x) = \left.\frac{\boxed{\substack{1\\2}}\cdot\boxed{\substack{2\\4}}}{\boxed{\substack{2\\3}}\cdot\left(\boxed{\substack{1\\2}}+\boxed{\substack{1\\3}}+\boxed{\substack{1\\4}}+\boxed{\substack{2\\3}}+\boxed{\substack{2\\4}}\right)}\right|_{v=x}\,,$$

$$b^{(3)}_3(x) = \left.\frac{\boxed{\substack{3\\4}}\cdot\left(\boxed{\substack{1\\2}}+\boxed{\substack{1\\3}}+\boxed{\substack{1\\4}}\right)}{\left(\boxed{\substack{2\\3}}+\boxed{\substack{2\\4}}\right)\cdot\left(\boxed{\substack{1\\2}}+\boxed{\substack{1\\3}}+\boxed{\substack{1\\4}}+\boxed{\substack{2\\3}}+\boxed{\substack{2\\4}}+\boxed{\substack{3\\4}}\right)}\right|_{v=x}\,, \tag{10}$$

where we have used an abbreviated notation using Young tableaux,

$$\boxed{j} = \lambda_j(v)\,,\qquad \boxed{\substack{j\\k}} = \lambda_j(v-\mathrm{i}/2)\lambda_k(v+\mathrm{i}/2)\,,$$
$$\boxed{\substack{j\\k\\l}} = \lambda_j(v-\mathrm{i})\lambda_k(v)\lambda_l(v+\mathrm{i})\,,\qquad (j\le k\le l)\,. \tag{11}$$

Unfortunately, no general scheme to construct these functions has been found so far. Hence, the auxiliary functions are only known for models with a fixed number of states $q\le 4$ including all possible gradings. For general models with $sl(n)$ symmetry, the number of necessary functions is conjectured to be the sum of the dimensions of all fundamental representations, i.e. 2^n-2. Only half as many functions, $2^{n-1}-1$, will be needed in the case of $sl(n|1)$-symmetric models.

Starting from the auxiliary functions $b^{(n)}_j(x)$ and $B^{(n)}_j(x) = b^{(n)}_j(x)+1$, for which the approximate positions of roots and poles are known from numerical calculations, the derivation is quite straightforward but lengthy. The key idea is to apply a Fourier transformation to the logarithmic derivative of all auxiliary functions and to exploit their analyticity properties in Fourier space. Eventually, relations among the auxiliary functions can be established, which form a set of coupled nonlinear integral equations of convolution type,

$$\ln b^{(n)}_j(x) = -\beta\left(V^{(n)}(x)+c^{(n)}_j\right) - \sum_{m,k}\left[\mathbf{K}^{(n,m)}_{j,k} * \ln B^{(m)}_k\right](x). \tag{12}$$

The driving terms $V^{(n)}(x)$, constants $c^{(n)}_j$ and matrices of kernel functions $\mathbf{K}^{(n,m)}(x)$ directly follow from the derivation. It is important to note that the Trotter number N appears explicitly only in the functions $V^{(n)}(x)$ for which the limit $N\to\infty$ can easily be taken analytically.

In our example, the $sl(4)$ case, the functions $V^{(n)}(x)$ and constants $c^{(n)}_j$ are

$$V^{(n)}(x) = \frac{\pi}{2}\frac{\sin(\pi n/4)}{\cosh(\pi x/2)-\cos(\pi n/4)}\,, \tag{13}$$

$$\begin{aligned}
c^{(1)}_1 &= (-3\mu_1+\mu_2+\mu_3+\mu_4)/4\,, & c^{(1)}_2 &= (\mu_1-3\mu_2+\mu_3+\mu_4)/4\,,\\
c^{(1)}_3 &= (\mu_1+\mu_2-3\mu_3+\mu_4)/4\,, & c^{(1)}_4 &= (\mu_1+\mu_2+\mu_3-3\mu_4)/4\,,\\
c^{(2)}_1 &= (-\mu_1-\mu_2+\mu_3+\mu_4)/2\,, & c^{(2)}_2 &= (-\mu_1+\mu_2-\mu_3+\mu_4)/2\,,\\
c^{(2)}_3 &= (-\mu_1+\mu_2+\mu_3-\mu_4)/2\,, & c^{(2)}_4 &= (\mu_1-\mu_2-\mu_3+\mu_4)/2\,,\\
c^{(2)}_5 &= (\mu_1-\mu_2+\mu_3-\mu_4)/2\,, & c^{(2)}_6 &= (\mu_1+\mu_2-\mu_3-\mu_4)/2\,,\\
c^{(3)}_1 &= (-\mu_1-\mu_2-\mu_3+3\mu_4)/4\,, & c^{(3)}_2 &= (-\mu_1-\mu_2+3\mu_3-\mu_4)/4\,,\\
c^{(3)}_3 &= (-\mu_1+3\mu_2-\mu_3-\mu_4)/4\,, & c^{(3)}_4 &= (3\mu_1-\mu_2-\mu_3-\mu_4)/4\,.
\end{aligned} \tag{14}$$

The kernel matrices $\mathbf{K}^{(n,m)}(x)$ are explicitly given by

$$\mathbf{K}^{(1,1)}(x) = \begin{pmatrix} K_0(x) & K_1(x) & K_1(x) & K_1(x)\\ K_2(x) & K_0(x) & K_1(x) & K_1(x)\\ K_2(x) & K_2(x) & K_0(x) & K_1(x)\\ K_2(x) & K_2(x) & K_2(x) & K_0(x)\end{pmatrix} = \mathbf{K}^{(3,3)}(x)\,,$$

$$\mathbf{K}^{(2,2)}(x) = \begin{pmatrix} K_3(x) & K_4(x) & K_4(x) & K_4(x) & K_4(x) & K_6(x)\\ K_5(x) & K_3(x) & K_4(x) & K_4(x) & K_8(x) & K_4(x)\\ K_5(x) & K_5(x) & K_3(x) & K_{10}(x) & K_4(x) & K_4(x)\\ K_5(x) & K_5(x) & K_{10}(x) & K_3(x) & K_4(x) & K_4(x)\\ K_5(x) & K_9(x) & K_5(x) & K_5(x) & K_3(x) & K_4(x)\\ K_7(x) & K_5(x) & K_5(x) & K_5(x) & K_5(x) & K_3(x)\end{pmatrix}\,,$$

$$\mathbf{K}^{(1,2)}(x) = \begin{pmatrix} K_{11}(x) & K_{11}(x) & K_{11}(x) & K_{12}(x) & K_{12}(x) & K_{12}(x)\\ K_{11}(x) & K_{14}(x) & K_{14}(x) & K_{11}(x) & K_{11}(x) & K_{12}(x)\\ K_{13}(x) & K_{11}(x) & K_{14}(x) & K_{11}(x) & K_{14}(x) & K_{11}(x)\\ K_{13}(x) & K_{13}(x) & K_{11}(x) & K_{13}(x) & K_{11}(x) & K_{11}(x)\end{pmatrix} = [\mathbf{K}^{(2,1)}(x)]^\dagger\,,$$

$$\mathbf{K}^{(1,3)}(x) = \begin{pmatrix} K_{15}(x) & K_{15}(x) & K_{15}(x) & K_{16}(x)\\ K_{15}(x) & K_{15}(x) & K_{18}(x) & K_{15}(x)\\ K_{15}(x) & K_{19}(x) & K_{15}(x) & K_{15}(x)\\ K_{17}(x) & K_{15}(x) & K_{15}(x) & K_{15}(x)\end{pmatrix} = [\mathbf{K}^{(3,1)}(x)]^\dagger\,,$$

$$\mathbf{K}^{(2,3)}_{j,k}(x) = \mathbf{K}^{(1,2)}_{5-k,7-j}(x)\,,\qquad \mathbf{K}^{(3,2)}(x) = [\mathbf{K}^{(2,3)}(x)]^\dagger\,, \tag{15}$$

where the kernels $K_j(x)$ are defined as $K_j(x) = \int_{-\infty}^{\infty}\widehat{K}_j(k)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k$ with

$$\begin{aligned}
\widehat{K}_0(k) &= \widehat{\mathcal{K}}^{(1,1)}(k)\,, & \widehat{K}_1(k) &= \widehat{\mathcal{K}}^{(1,1)}(k)+\mathrm{e}^{-k/2-|k|/2}\,,\\
\widehat{K}_2(k) &= \widehat{\mathcal{K}}^{(1,1)}(k)+\mathrm{e}^{k/2-|k|/2}\,, & \widehat{K}_3(k) &= \widehat{\mathcal{K}}^{(2,2)}(k)\,,\\
\widehat{K}_4(k) &= \widehat{\mathcal{K}}^{(2,2)}(k)+\mathrm{e}^{-k/2-|k|/2}\,, & \widehat{K}_5(k) &= \widehat{\mathcal{K}}^{(2,2)}(k)+\mathrm{e}^{k/2-|k|/2}\,,\\
\widehat{K}_6(k) &= \widehat{\mathcal{K}}^{(2,2)}(k)+\mathrm{e}^{-k-|k|}\,, & \widehat{K}_7(k) &= \widehat{\mathcal{K}}^{(2,2)}(k)+\mathrm{e}^{k-|k|}\,,\\
\widehat{K}_8(k) &= \widehat{\mathcal{K}}^{(2,2)}(k)+2\mathrm{e}^{-k/2-|k|/2}\,, & \widehat{K}_9(k) &= \widehat{\mathcal{K}}^{(2,2)}(k)+2\mathrm{e}^{k/2-|k|/2}\,,\\
\widehat{K}_{10}(k) &= \widehat{\mathcal{K}}^{(2,2)}(k)+\mathrm{e}^{-|k|}\,, & \widehat{K}_{11}(k) &= \widehat{\mathcal{K}}^{(1,2)}(k)\,,\\
\widehat{K}_{12}(k) &= \widehat{\mathcal{K}}^{(1,2)}(k)+\mathrm{e}^{-k-|k|/2}-\mathrm{e}^{-k/2}\,, & \widehat{K}_{13}(k) &= \widehat{\mathcal{K}}^{(1,2)}(k)+\mathrm{e}^{k-|k|/2}-\mathrm{e}^{k/2}\,,\\
\widehat{K}_{14}(k) &= \widehat{\mathcal{K}}^{(1,2)}(k)+\mathrm{e}^{-|k|/2}\,, & \widehat{K}_{15}(k) &= \widehat{\mathcal{K}}^{(1,3)}(k)\,,\\
\widehat{K}_{16}(k) &= \widehat{\mathcal{K}}^{(1,3)}(k)+\mathrm{e}^{-3k/2-|k|/2}-\mathrm{e}^{-k}\,, & \widehat{K}_{17}(k) &= \widehat{\mathcal{K}}^{(1,3)}(k)+\mathrm{e}^{3k/2-|k|/2}-\mathrm{e}^{k}\,,\\
\widehat{K}_{18}(k) &= \widehat{\mathcal{K}}^{(1,3)}(k)+\mathrm{e}^{-k/2-|k|/2}-1\,, & \widehat{K}_{19}(k) &= \widehat{\mathcal{K}}^{(1,3)}(k)+\mathrm{e}^{k/2-|k|/2}.
\end{aligned} \tag{16}$$

The function $\widehat{\mathcal{K}}^{(n,m)}(k)$ is related to the S-matrix and is given by

$$\widehat{\mathcal{K}}^{(n,m)}(k) = \mathrm{e}^{|k|/2}\frac{\sinh(\min(n,m)k/2)\sinh([4-\max(n,m)]k/2)}{\sinh(k/2)\sinh(2k)} - \delta_{n,m}\,. \tag{17}$$

We note that in spectral parameter space all kernel functions can be written in terms of digamma and simple rational functions.

Finally, the largest eigenvalue can be recovered from the auxiliary functions,

$$\ln\Lambda_{\max}(0) = -\beta e_0 + \sum_{n,j}\left[V^{(n)} * \ln B^{(n)}_j\right](0)\,, \tag{18}$$

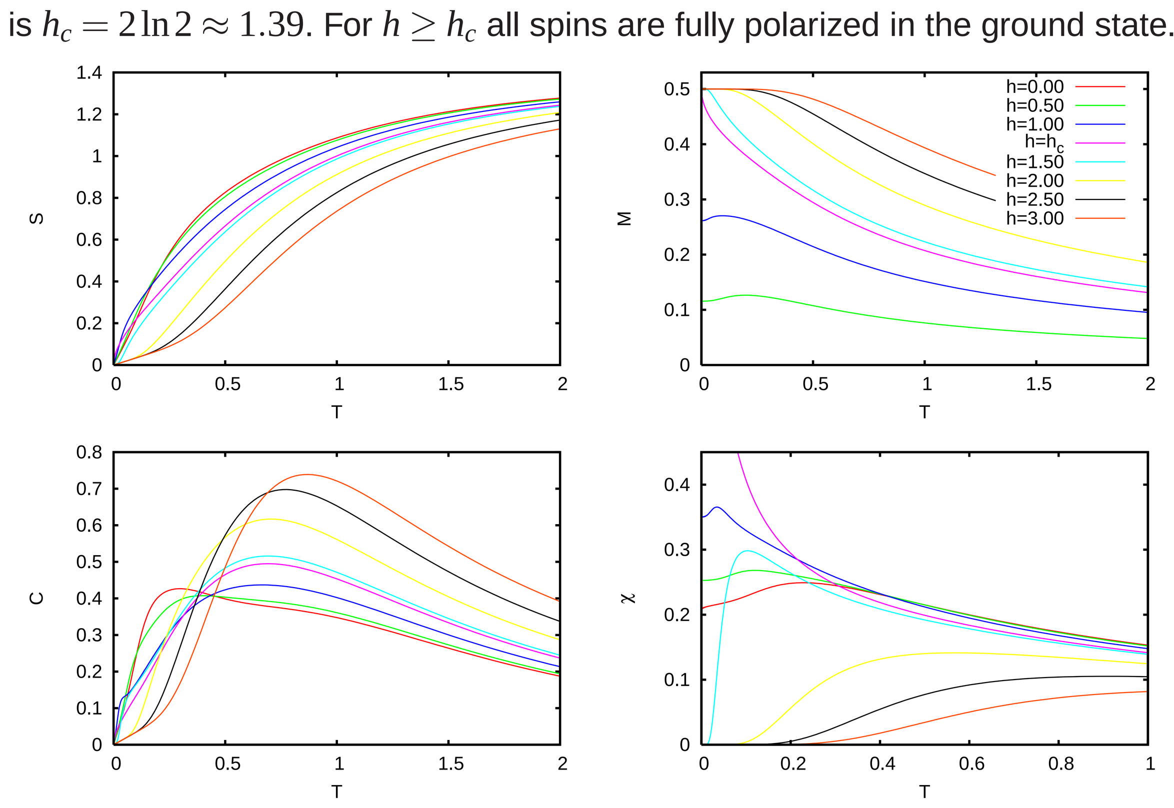
where e_0 is the ground-state energy, which also follows from the previous calculations. For the $sl(4)$ case, we get $e_0 = 1-\pi/4-3\ln(2)/2-\sum_{j=1}^{4}\mu_j/4$.

Therefore, the problem of solving the infinitely many BA equations (8) in the limit $N\to\infty$ has been reduced to finding a finite set of functions satisfying the NLIE (12). The NLIE are valid for arbitrary finite temperature and chemical potentials. In particular, the evaluation at low temperature poses no problem.

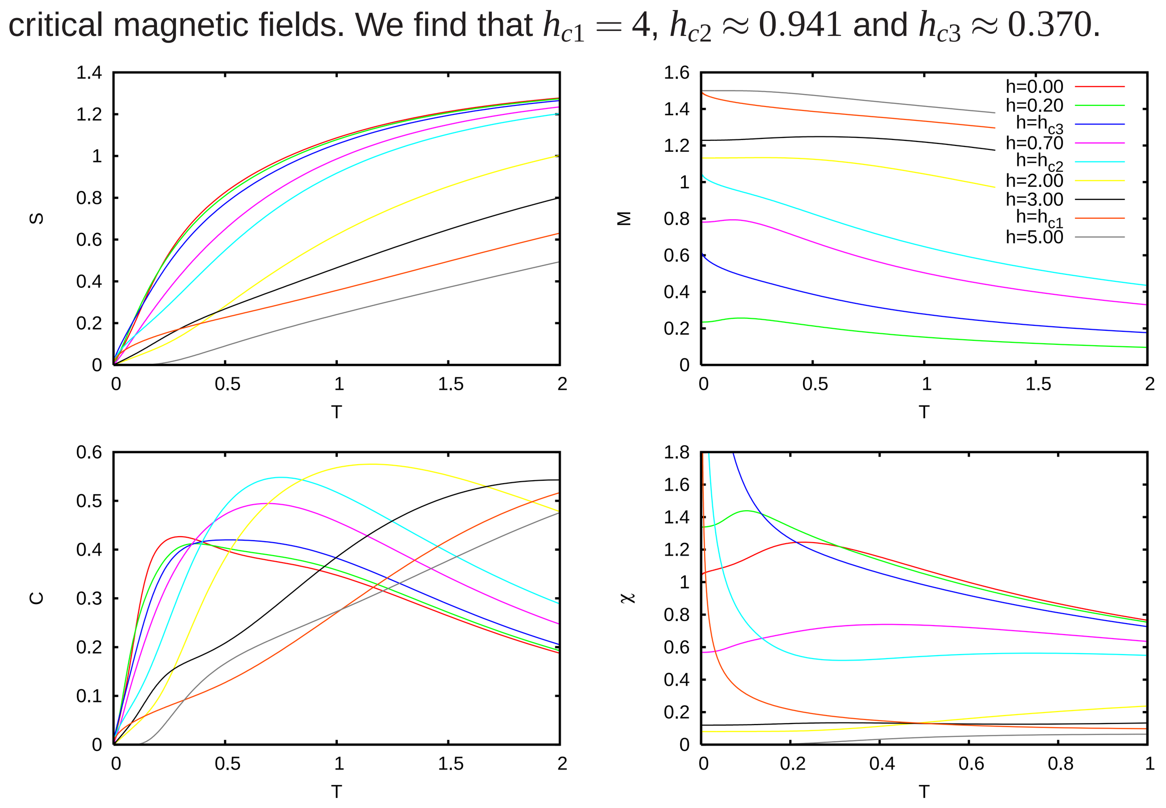
We like to note that there exists yet another type of NLIE, which have already been generalized for Uimin-Sutherland models of arbitrary type [2]. However, these prove difficult to evaluate at low temperature. Only high-temperature expansions, albeit to a very high degree (~ 40), have been obtained in that way.

4 Numerical results

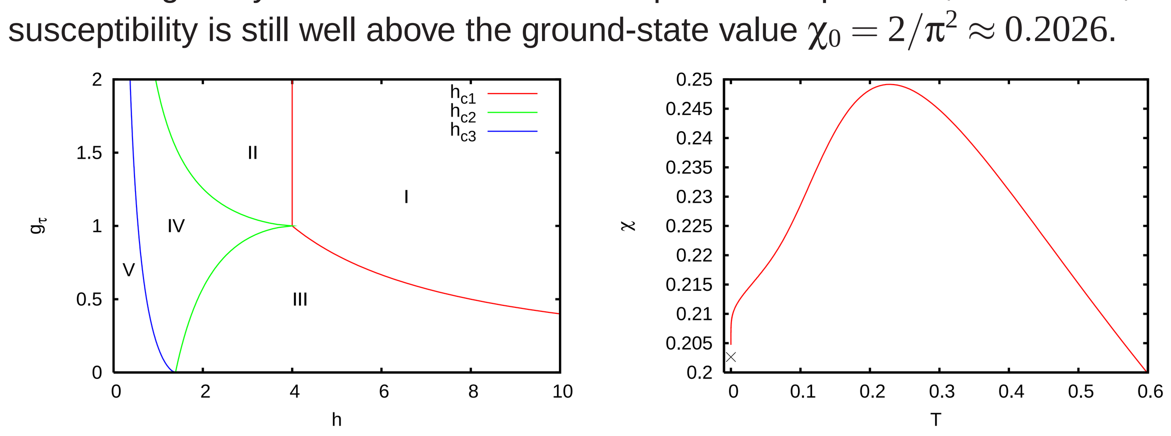
As the NLIE (12) are of convolution type, they are well suited for a numerical solution by iteration utilizing the fast Fourier transform algorithm. In the following, we present some results for the $SU(4)$ spin-orbital model (9). The entropy S, specific heat C, magnetization M and magnetic susceptibility χ in the case $g_S = 1$, $g_\tau = 0$ for various magnetic fields are shown below. The critical field is $h_c = 2\ln 2 \approx 1.39$. For $h \ge h_c$ all spins are fully polarized in the ground state.



We also present results for the case $g_S = 1$, $g_\tau = 2$. In this case there are three critical magnetic fields. We find that $h_{c1} = 4$, $h_{c2} \approx 0.941$ and $h_{c3} \approx 0.370$.



The critical fields follow directly from the $T\to 0$ limit of our NLIE. The phase diagram for fixed $g_S = 1$ is shown below on the left. The figure on the right shows the zero field susceptibility for the case $g_S = 1$, $g_\tau = 0$ featuring a typical logarithmic singularity at $T = 0$. For the lowest plotted temperature, $T = 10^{-10}$, the susceptibility is still well above the ground-state value $\chi_0 = 2/\pi^2 \approx 0.2026$.



References

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[2] Z. Tsuboi, *Nonlinear integral equations and high temperature expansion for the $U_q(\widehat{sl}(r+1|s+1))$ Perk-Schultz model*, Nucl. Phys. B **737** (2006), 261.