Nonlinear Integral Equations for the Thermodynamics of Integrable Quantum Chains

Dissertation

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vorgelegt von

Jens Damerau

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Chapter 1

Introduction

Theoretical many-particle physics aims to describe the behaviour of systems involving large numbers of interacting particles. This poses a difficult problem even if the elementary interactions are known, since the total degree of freedom increases exponentially with the number of involved particles. A thermodynamical description, which relies on the knowledge of all eigenstates and eigenvalues of the quantum mechanical system, therefore usually requires the application of approximations, where the multi-particle interactions are replaced by effective single-particle terms, for instance by averaging the mutual interactions. This mean field approach is, however, known to be problematic in the vicinity of phase transitions, where the behaviour of a system is governed by collective phenomena.

Although there exists a well-defined notion of integrability in classical mechanics due to Liouville's theorem, no such general notion is known for quantum mechanical systems. Still, certain classes of systems have been found that admit an exact solution. The first example dates back to 1931, when Bethe invented a method to obtain the complete spectrum of the isotropic spin-1/2 Heisenberg chain [12]. He could show that each eigenvalue of the Hamiltonian for a chain of length N may be described with the help of just a finite set of $M \leq N/2$ many complex numbers, the Bethe ansatz roots, which are completely determined by M coupled algebraic equations. Although Bethe's original plan to further generalize his approach—nowadays known as the coordinate Bethe ansatz—to allow for spatial lattices proved to be too optimistic, his seminal work paved the way for the subsequent solution of many other related one-dimensional quantum models, thus constituting the new field of integrable quantum chains.

As the first alternative application of the Bethe ansatz, the spectrum of the Bose gas with delta-function interactions could be obtained by Lieb and Liniger [62]. Lieb furthermore extended the ansatz to cope with certain two-dimensional problems from statistical mechanics, namely special cases of the six-vertex model [59–61]. The general solution of the six-vertex model was then given by Sutherland [73]. Another important refinement of the Bethe ansatz technique is due to the work of Yang [97] and Gaudin [29], who managed to deal with systems including internal degrees of freedom with the help of a nested Bethe ansatz.

It was soon realized that the exact solvability of quantum chains relies on the factorization of the many-body scattering matrix into a product of scattering matrices of just two particles. This does only happen, if the corresponding two-particle scattering matrices satisfy a self-consistency relation known as the Yang-Baxter equation. Remarkably, the same equation also applies to the local Boltzmann weights of the solvable two-dimensional classical models, where it gives rise to a commuting family of transfer matrices. As a consequence of this mutual relationship between both types of models, it follows that the logarithmic derivative of the transfer matrix of a given solvable two-dimensional model—taken at some special spectral parameter—also defines the Hamiltonian of an integrable quantum chain. For a review, see Baxter's book [10] and references therein. Note that Baxter also showed that the star-triangle relations that have been earlier used in Onsager's solution of the two-dimensional Ising model [64] are in fact equivalent to the Yang-Baxter equation.

The discovery of the Yang-Baxter equation as the defining feature of integrable models soon lead to the development of the so-called algebraic Bethe ansatz, where, based on a given solution called R matrix, purely algebraic techniques are used for the construction of the eigenvalues and eigenstates of the transfer matrix and thereby also of the Hamiltonian of the corresponding quantum model, see [49] for a review. Moreover, it initiated a systematic search for new solutions [51, 52] eventually leading to the discovery of quantum groups [21, 33].

The first treatment of the thermodynamical properties of an integrable model has been achieved by Yang and Yang for the case of the delta-function Bose gas [98, 99]. Later, the method—known as the thermodynamic Bethe ansatz (TBA)—was independently extended by Gaudin [30] and Takahashi [82] in order to treat the thermodynamics of the spin-1/2 Heisenberg model, see [83] for a review. In their work, they exploit the observation that the Bethe ansatz roots corresponding to excited states of the model form certain string patterns in the complex plane. Admitting these distributions in general, which is known as the string hypothesis, they are able to classify all excitations and calculate the corresponding eigenvalues. The thermodynamic limit of the free energy is finally found to be encoded by a set of infinitely many nonlinear integral equations (NLIEs).

An alternative method to calculate the free energy that does not rely on the string hypothesis has later been introduced by Klümper [40, 41]. He starts with a Trotter-Suzuki mapping [78–80, 87] in order to express the partition function of the one-dimensional quantum chain by that of an equivalent inhomogeneous two-dimensional classical model. The latter is then expressed in terms of the eigenvalues of an adapted quantum transfer matrix (QTM), where only the largest eigenvalue survives in the thermodynamic limit [48]. With the help of an assumption on the analyticity properties of two suitable auxiliary functions, which are backed by numerics, he finally manages to encode the largest eigenvalue of the QTM—and thus the free energy—by a set of just two coupled NLIEs. Remarkably, these equations turn out to be extremely well posed for numerical evaluations at arbitrary temperature and magnetic field. Note that similar equations have previously been established for the treatment of finite size corrections [43, 44].

There exists yet another type of NLIEs which has been discovered by Takahashi [84] in an attempt to simplify the TBA equations. In his work, he finally obtains just a single

NLIE for the largest eigenvalue of the QTM. Although it is difficult to numerically evaluate the equation for finite temperature, high-temperature expansions have been achieved up to very high order [71]. Note that his NLIE has also been rederived on the basis of the QTM approach [85].

To this day, many interesting models could be solved by a combination of the available techniques, see [23] for example. Recent developments even allow for the derivation of finite-temperature correlation functions for the case of the XXZ chain, see [15] and references therein.

Let us now turn to the main subject of our work, the Uimin-Sutherland model. In 1970 Uimin proposed and solved a three-component generalization of the spin-1/2 Heisenberg model in order to deal with a spin-1 chain [93]. Sutherland then showed that the general multi-component, higher-rank generalization of this model is also solvable by the nested Bethe ansatz and managed to calculate its ground-state energy and excitations [74]. The Hamiltonian is given as a sum over nearest-neighbour permutators, where Sutherland additionally introduces a $\mathbb{Z}_2$ grading to allow for fermionic as well as bosonic components. The two-dimensional classical model associated to the Hamiltonian of the Uimin-Sutherland model was later discovered to be a special case of the Perk-Schultz model [65, 69], which itself is a generalization of the six-vertex model. The models correspond to the fundamental representation of the supersymmetry algebra $sl(r|s)$, see [16] for a review on supersymmetries. The case of higher-level representations of the underlying symmetry algebra has been treated by Andrei and Johannesson [4, 35]. Affleck was the first to calculate the critical behaviour based on non-Abelian bosonization and conformal field theory [1, 3].

Despite its simplicity, the model has many interesting applications to integrable quantum chains arising from several contexts. Besides the spin-1/2 Heisenberg chain and the three-component, $sl(3)$-symmetric case investigated by Uimin, models that are known to fall in this class are the supersymmetric t-J model [67], the $SU(4)$ spin-orbital model [95, 96], its $SU(4|1)$ generalization including mobile defects, the Essler-Korepin-Schoutens model [24, 25] and certain spin-ladder systems [6, 94].

Using the TBA approach for the thermodynamics of the $sl(n)$-symmetric Uimin-Sutherland model [34], the low-field asymptotics of the susceptibility [68] as well as the low-temperature asymptotics of the specific heat [56, 57] have been derived analytically. TBA equations for the general $sl(r|s)$-symmetric case have been provided by Saleur [66]. Note that it has been shown by Jüttner, Klümper and Suzuki that the final TBA equations may as well be derived in the context of the QTM formalism without using the string hypothesis [38]. Instead, they exploit the functional relations provided by the fusion hierarchy of transfer matrices [53, 88, 89] and additionally use certain assumptions on the analyticity properties of the fused eigenvalue functions, which are backed by numerics. Unfortunately, the numerical evaluation of the infinitely many coupled TBA equations generally poses a problem as some kind of truncation scheme is necessary.

An alternative approach by generalizing Takahashi's single NLIE has been pursued by Tsuboi [90]. He managed to derive a finite set of coupled NLIEs, where the number of equations is equal to the rank of the underlying algebra. Still, the numerical evaluation of his NLIEs turns out to be difficult, since their structure involves integration contours along rather complicated complex paths. Similar to Takahashi's NLIE however, the formulation admits high-temperature expansions up to very high orders (~ 40) [71, 92]. Recently, Tsuboi managed to further extend the approach to deal with the general $U_q(\widehat{sl}(r|s))$-symmetric Perk-Schultz model [91].

However, since neither the TBA nor Tsuboi's equations allow for a precise numerical evaluation in the interesting low-temperature regime, it is still worthwhile to search for finite sets of NLIEs in the spirit of [41], which are known to yield accurate numerical results for arbitrary finite temperature. Unfortunately, no straightforward way of getting the required auxiliary functions, from which this type of NLIEs can be derived, is known. Previously, such auxiliary functions have been known only for cases involving three components at most [27, 36, 37].

The goal of our work has been to further investigate the structure of the latter type of NLIEs and to extend the approach to new subcases of the Uimin-Sutherland model. Although a general construction of the necessary auxiliary functions stays out of reach, we are able to provide suitable auxiliary functions for all possible four-component cases and derive their corresponding NLIEs. The auxiliary functions are shown to be intimately linked to those of the TBA approach. Based on the observed general structure, we conjecture the final NLIEs of two five-component NLIEs. Moreover, we derive accurate numerical results for several interesting applications.

The outline of this thesis is as follows. In Chapter 2, we briefly introduce the general Uimin-Sutherland model, mention several of its applications to integrable quantum chains and review the QTM approach to the thermodynamics of the model. In Chapter 3, we start with a review of the fusion hierarchy relations. Thereafter, we show how the set of relations can be exploited in order to derive the TBA equations for the Uimin-Sutherland model. Chapter 4 deals with the derivation of the finite sets of Klümper-type NLIEs. We start with a review of the previous work in order to unify the notation and to introduce some additional observations. After this, we present new sets of auxiliary functions from which we obtain well-posed coupled sets of NLIEs for all four-component cases of the Uimin-Sutherland model. Note that these are the main results of this thesis. Several limiting cases of the NLIEs are considered to check the validity of the results. Finally, we show how the Klümper-type auxiliary functions can be modified to exactly truncate the NLIEs obtained from the TBA approach at an arbitrary level. In contrast to these rigorous results, Chapter 5 is devoted to conjectures for higher-rank cases of the Uimin-Sutherland model. Based on some reasonable assumptions on the general structure of the NLIEs and by exploiting certain limiting cases, we are able to find the final form of the NLIEs for the $sl(5)$- and $sl(4|1)$-symmetric cases. Furthermore, we conjecture the general form of the exactly truncated TBA equations. In Chapter 6, we deal with the efficient numerical evaluation of the NLIEs. The validity and accuracy of our results is checked by a comparison with Tsuboi's high-temperature expansions of the specific heat. After this, we present various new results for the applications of the Uimin-Sutherland model. Finally, in Chapter 7, we give a summary of our work and an outlook on open problems and possible future developments.

The appendices cover material that has been deferred from the main text. In Appendix A, we show how to translate between the Bethe ansatz results for equivalent grading choices.

Appendix B contains the derivation of the TBA equations for the $sl(n|1)$-symmetric case of the Uimin-Sutherland model. Similarly, Appendix C contains the explicit derivation of the Klümper-type NLIEs for the $sl(4)$-symmetric case of the Uimin-Sutherland model. Appendix D introduces some observations on the general structure of the kernel matrix which are somehow related to the underlying algebra. Finally, the explicit kernel matrix of the $sl(5)$-symmetric case is given in Appendix E.

Note that two publications [18] and [17] are more or less based on the work on this thesis. The former of these deals with the new sets of auxiliary functions and NLIEs for the $sl(4)$-symmetric case of the Uimin-Sutherland model, confer Section 4.2.1. In the latter publication, which deals with the correlation functions of the spin-1/2 Heisenberg chain at finite lengths, merely the same software implementation for solving NLIEs has been used to generate the numerical results. The topic is nevertheless not part of this thesis.

Chapter 2

Thermodynamics of the Uimin-Sutherland model

In this chapter, we begin with a short review of the Uimin-Sutherland model [74, 93] and introduce some of its applications to specific integrable quantum chains. Thereafter, we briefly discuss the quantum transfer matrix approach to the thermodynamics of the model, which will be the basis for the derivation of the nonlinear integral equations in the successive chapters.

2.1 Definition of the Uimin-Sutherland model

The Uimin-Sutherland model is defined on a one-dimensional lattice with L sites, where a q-state spin-variable α_j is assigned to each site j. The local basis is $\mathbb{Z}_2$ graded, where the grading of each spin α will be denoted by $p(\alpha)$. The global basis is then obtained by a tensor product of the local spins,

$$|\alpha_1 \, \ldots \, \alpha_L\rangle = |\alpha_1\rangle \otimes \cdots \otimes |\alpha_L\rangle \, . \tag{2.1}$$

The Hamiltonian of the Uimin-Sutherland model is given by

$$\mathcal{H}_0 = J \sum_{j=1}^{L} \pi_{j,j+1} \, , \tag{2.2}$$

where the local interaction operators $\pi_{j,j+1}$ permute neighbouring spins on the lattice with respect to their grading,

$$\pi_{j,j+1}|\ldots \, \alpha_j \, \alpha_{j+1} \, \ldots\rangle = (-1)^{p(\alpha_j)p(\alpha_{j+1})}|\ldots \, \alpha_{j+1} \, \alpha_j \, \ldots\rangle \, . \tag{2.3}$$

Note that periodic boundary conditions are imposed, so that $\alpha_{L+1} = \alpha_1$. Accordingly, the total number of spins of each type α is conserved. Moreover, the model shows $sl(r|s)$ symmetry, where r and s ($q = r + s$) are the total number of states with grading $p(\alpha) = 0$ and $p(\alpha) = 1$, respectively.

The one-dimensional Uimin-Sutherland model is known to be exactly solvable by Bethe ansatz [74]. Its classical two-dimensional counterpart is given by the Perk-Schultz model [65, 69], which will be introduced in Section 2.3.1.

In order to incorporate the effect of external fields, we may add certain additional terms to the Hamiltonian (2.2),

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{ext}} = \mathcal{H}_0 - \sum_{j=1}^{L} \sum_{\alpha=1}^{q} \mu_\alpha n_{j,\alpha} \,, \tag{2.4}$$

where μ_α is some general chemical potential associated with state α and the operator $n_{j,\alpha}$ counts the number of particles of type α sitting on site j. Although terms of this type generally break the $sl(r|s)$ symmetry, we will see in the following that they do not affect the integrability of the thermodynamical properties. Note that these terms play a crucial role in order to deal with the applications of the model.

2.2 Applications to integrable quantum chains

Several interesting integrable quantum chains are of Uimin-Sutherland type. In the following, we will briefly mention some of the possible applications.

2.2.1 The spin-1/2 Heisenberg chain

Since the Uimin-Sutherland model is basically a higher-rank generalization of the one-dimensional spin-1/2 Heisenberg chain,

$$\mathcal{H} = J \sum_{j=1}^{L} (2\boldsymbol{S}_j \boldsymbol{S}_{j+1} + 1/2) - h \sum_{j=1}^{L} S_j^z \,, \tag{2.5}$$

this model is of course contained as a special case. Using the two-state basis

$$|1\rangle = |\uparrow\rangle \,, \qquad |2\rangle = |\downarrow\rangle \,, \tag{2.6}$$

with $p(1) = p(2) = 0$, it is easy to see that the first term is equivalent to that of the corresponding Uimin-Sutherland model. Since the external field term is diagonal in the given basis, it can also be treated within the framework of the Uimin-Sutherland model by choosing the general chemical potentials to be

$$\mu_1 = h/2 \,, \qquad \mu_2 = -h/2 \,. \tag{2.7}$$

2.2.2 The spin-1 chain with biquadratic interaction term

Let us now turn to the three-state, $sl(3)$-symmetric case of the Uimin-Sutherland model, which is the simplest higher rank generalization. Identifying the three ungraded basis states ($p(\alpha) = 0$ for $\alpha = 1, 2, 3$) with the eigenstates of the spin-1 S_z-operator, we find that this case is connected to some spin-1 Heisenberg chain with an additional biquadratic interaction term [93]

$$\mathcal{H} = J \sum_{j=1}^{L} \left\{ \boldsymbol{S}_j \boldsymbol{S}_{j+1} + (\boldsymbol{S}_j \boldsymbol{S}_{j+1})^2 \right\} - h \sum_{j=1}^{L} S_j^z \,. \tag{2.8}$$

Again we have introduced an external magnetic field term, which according to our basis choice, leads to the general chemical potentials

$$\mu_1 = h \,, \qquad \mu_2 = 0 \,, \qquad \mu_3 = -h \,. \tag{2.9}$$

Note that the Hamiltonian of this model is strongly related to another exactly solvable model, the Takhtajan-Babujian model [5, 86], where just the sign in front of the biquadratic term is changed.

2.2.3 The supersymmetric *t-J* model

Consider the Hamiltonian of the one-dimensional t-J model [67],

$$\begin{aligned} \mathcal{H} = t \sum_{j=1}^{L} \sum_{\sigma=\uparrow,\downarrow} \mathcal{P} \left(c_{j,\sigma}^\dagger c_{j+1,\sigma} + c_{j,\sigma} c_{j+1,\sigma}^\dagger \right) \mathcal{P} + J \sum_{j=1}^{L} \left(\boldsymbol{S}_j \boldsymbol{S}_{j+1} - n_j n_{j+1}/4 \right) \\ - \mu \sum_{j=1}^{L} n_j - h \sum_{j=1}^{L} S_j^z \,. \end{aligned} \tag{2.10}$$

The origin of this model is the strong-coupling limit of the Hubbard model, where only the nearest-neighbour interaction terms are retained. It is also a natural generalization of the spin-1/2 Heisenberg chain in order to incorporate mobile defects. There are three possible basis states per lattice site: Each site may be empty, or there may be an electron with either spin up or down. Double occupancy is forbidden due to strong on-site repulsion. The first term of the Hamiltonian describes the hopping of electrons, while the second term deals with the spin interactions. The operators $c_{j,\sigma}$ and $c_{j,\sigma}^\dagger$ annihilate and create, respectively, an electron at site j with spin σ, the projector $\mathcal{P} = \prod_{j=1}^{L} (1 - n_{j,\uparrow} n_{j,\downarrow})$ inhibits the occurrence of two electrons at the same site, and n_j counts the number of particles on site j.

Note, however, that in general the model is neither integrable nor of Uimin-Sutherland type. Only two special cases, $J = 0$ and $J = 2t$, are integrable, and only the latter of these, the so-called supersymmetric case of the t-J model can be treated within the Uimin-Sutherland framework. It is found to be equivalent to the $sl(2|1)$-symmetric case, where one of the three basis states is graded.

Let us fix the grading $p(1) = p(2) = 0$ and $p(3) = 1$ and accordingly identify the basis states

$$|1\rangle = |\uparrow\rangle \,, \qquad |2\rangle = |\downarrow\rangle \,, \qquad |3\rangle = |0\rangle \,, \tag{2.11}$$

where $|0\rangle$ denotes the empty site. Using these definitions, the Hamiltonian (2.10) with $J = 2t$ may be cast as

$$\mathcal{H} = t \sum_{j=1}^{L} \pi_{j,j+1} + tL - (2t+\mu) \sum_{j=1}^{L} n_j - h \sum_{j=1}^{L} S_j^z \,, \tag{2.12}$$

where the first term is exactly the Hamiltonian $\mathcal{H}_0$ of the $sl(2|1)$-symmetric case of the Uimin-Sutherland model (2.2). Because the remaining terms are already diagonal in the given basis, they are also compatible with the Uimin-Sutherland formulation. By comparison with equation (2.4), we find

$$\mu_1 = t + \mu + h/2 \,, \qquad \mu_2 = t + \mu - h/2 \,, \qquad \mu_3 = -t \,. \tag{2.13}$$

2.2.4 The $SU(4)$ spin-orbital model

Another special case that we want to mention is a generalization of the spin-1/2 Heisenberg model, where each electron carries an orbital pseudospin in addition to the spin [95, 96]. We have the Hamiltonian

$$\mathcal{H} = J \sum_{j=1}^{L} (2\boldsymbol{S}_j \boldsymbol{S}_{j+1} + 1/2)(2\boldsymbol{\tau}_j \boldsymbol{\tau}_{j+1} + 1/2) - h \sum_{j=1}^{L} \left(g_S S_j^z + g_\tau \tau_j^z\right) \,, \tag{2.14}$$

where $\boldsymbol{S}_j$ and $\boldsymbol{\tau}_j$ are the spin and pseudospin operators, respectively. The Landé factors g_S and g_τ adjust the respective coupling strengths to the external magnetic field. This model corresponds to the four-state, $sl(4)$-symmetric case of the Uimin-Sutherland, where we use the basis

$$|1\rangle = |{\uparrow_S}{\uparrow_\tau}\rangle \,, \qquad |2\rangle = |{\uparrow_S}{\downarrow_\tau}\rangle \,, \qquad |3\rangle = |{\downarrow_S}{\uparrow_\tau}\rangle \,, \qquad |4\rangle = |{\downarrow_S}{\downarrow_\tau}\rangle \tag{2.15}$$

and accordingly set the general chemical potentials

$$\mu_1 = (g_S + g_\tau)h/2 \,, \qquad \mu_2 = (g_S - g_\tau)h/2 \,, \tag{2.16a}$$

$$\mu_3 = -(g_S - g_\tau)h/2 \,, \qquad \mu_4 = -(g_S + g_\tau)h/2 \tag{2.16b}$$

for the inclusion of the magnetic field term.

2.2.5 The two-leg spin-1/2 ladder

Yet another possible application of the $sl(4)$-symmetric Uimin-Sutherland model is a certain two-leg spin-1/2 ladder [94],

$$\mathcal{H} = J_\parallel \mathcal{H}_{\text{leg}} + J_\perp \sum_{j=1}^{L} \boldsymbol{S}_j \boldsymbol{T}_j - h \sum_{j=1}^{L} \left(S_j^z + T_j^z\right) \,, \tag{2.17}$$

where

$$\mathcal{H}_{\text{leg}} = \sum_{j=1}^{L} \left(\boldsymbol{S}_j \boldsymbol{S}_{j+1} + \boldsymbol{T}_j \boldsymbol{T}_{j+1} + 4\left(\boldsymbol{S}_j \boldsymbol{S}_{j+1}\right)\left(\boldsymbol{T}_j \boldsymbol{T}_{j+1}\right)\right) \,. \tag{2.18}$$

Here, $\boldsymbol{S}_j$ and $\boldsymbol{T}_j$ are the spin operators for the two legs of the ladder. The intrachain and rung couplings are controlled by the coupling constants $J_\parallel$ and $J_\perp$, respectively. Apart from the appearance of the biquadratic interaction term in $\mathcal{H}_{\text{leg}}$, this model is equal to the usual two-leg Heisenberg ladder, which, however, is not integrable. The Hamiltonian $\mathcal{H}_{\text{leg}}$ may alternatively be written as

$$\mathcal{H}_{\text{leg}} = \sum_{j=1}^{L} (2\boldsymbol{S}_j\boldsymbol{S}_{j+1} + 1/2)\,(2\boldsymbol{T}_j\boldsymbol{T}_{j+1} + 1/2) - L/4\,, \tag{2.19}$$

which, up to the constant $L/4$, is equivalent to the four-state Uimin-Sutherland Hamiltonian. In order to include the remaining terms, we have to choose a basis, in which the interchain coupling is diagonal. This is accomplished by the basis

$$|1\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)\,, \qquad |2\rangle = |\uparrow\uparrow\rangle\,, \tag{2.20a}$$

$$|3\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)\,, \qquad |4\rangle = |\downarrow\downarrow\rangle\,. \tag{2.20b}$$

Note that the state $|1\rangle$ is a singlet state with respect to the intrachain coupling, while the remaining states form a rung triplet. Accordingly, the general chemical potential of the Uimin-Sutherland model are

$$\mu_1 = (J_\parallel + 3J_\perp)/4\,, \qquad \mu_2 = (J_\parallel - J_\perp)/4 + h\,, \tag{2.21a}$$

$$\mu_3 = (J_\parallel - J_\perp)/4\,, \qquad \mu_4 = (J_\parallel - J_\perp)/4 - h\,. \tag{2.21b}$$

So, this model can also be treated within the framework of the Uimin-Sutherland model.

2.2.6 The Essler-Korepin-Schoutens model

The next application we are going to deal with is the Essler-Korepin-Schoutens model [24, 25]. The model is similar to the supersymmetric t-J model, but here double occupancy is allowed at each site and controlled by a Hubbard interaction term. Thus the model combines features of both the t-J and the Hubbard model. The Hamiltonian is given by

$$\mathcal{H} = J\mathcal{H}_0 + U\sum_{j=1}^{L}(n_{j,\uparrow} - 1/2)(n_{j,\downarrow} - 1/2) - \mu\sum_{j=1}^{L} n_j\,, \tag{2.22}$$

where

$$\begin{aligned}\mathcal{H}_0 = \sum_{j=1}^{L} \Big\{ & (c_{j,\uparrow}^\dagger c_{j+1,\uparrow} + c_{j,\uparrow} c_{j+1,\uparrow}^\dagger)(1 - n_{j,\downarrow} - n_{j+1,\downarrow}) \\ & + (c_{j,\downarrow}^\dagger c_{j+1,\downarrow} + c_{j,\downarrow} c_{j+1,\downarrow}^\dagger)(1 - n_{j,\uparrow} - n_{j+1,\uparrow}) \\ & + (n_j - 1)(n_{j+1} - 1)/2 - (n_{j,\uparrow} - n_{j,\downarrow})(n_{j+1,\uparrow} - n_{j+1,\downarrow})/2 \\ & + c_{j,\uparrow}^\dagger c_{j,\downarrow}^\dagger c_{j+1,\downarrow} c_{j+1,\uparrow} + c_{j,\uparrow} c_{j,\downarrow} c_{j+1,\downarrow}^\dagger c_{j+1,\uparrow}^\dagger \\ & - c_{j,\downarrow}^\dagger c_{j,\uparrow} c_{j+1,\uparrow}^\dagger c_{j+1,\downarrow} - c_{j,\uparrow}^\dagger c_{j,\downarrow} c_{j+1,\downarrow}^\dagger c_{j+1,\uparrow} \\ & + (n_{j,\uparrow} - 1/2)(n_{j,\downarrow} - 1/2) + (n_{j+1,\uparrow} - 1/2)(n_{j+1,\downarrow} - 1/2) \Big\} . \end{aligned} \tag{2.23}$$

Despite the lengthy form of $\mathcal{H}_0$, all of its L many summands are nothing else but graded permutation operators $\pi_{j,j+1}$ of nearest-neighbour states. Let us fix the basis

$$|1\rangle = |\uparrow\rangle , \qquad |2\rangle = |\downarrow\rangle , \tag{2.24a}$$

$$|3\rangle = |\mathrm{d}\rangle , \qquad |4\rangle = |0\rangle , \tag{2.24b}$$

where $|\mathrm{d}\rangle$ denotes a doubly occupied site. The grading for this particular choice is $p(1) = p(2) = 0$ and $p(3) = p(4) = 1$. The model therefore corresponds to the $sl(2|2)$-symmetric case of the Uimin-Sutherland model. Both the Hubbard and the chemical potential terms are diagonal in this basis and can thus be treated within the framework of the general chemical potentials,

$$\mu_1 = U/4 + \mu + h/2 , \qquad \mu_2 = U/4 + \mu - h/2 , \tag{2.25a}$$

$$\mu_3 = -U/4 + 2\mu , \qquad \mu_4 = -U/4 . \tag{2.25b}$$

2.2.7 The $SU(4|1)$ spin-orbital model with mobile defects

Finally, let us consider a model that serves both as a generalization of the supersymmetric t-J model and the $SU(4)$ spin-orbital model. It arises either by introducing an additional pseudospin-1/2 degree of freedom for each electron in the supersymmetric t-J model or, equivalently, by adding mobile empty sites to the $SU(4)$ spin-orbital model. Note that the model is similar to [58]. The Hamiltonian is given by

$$\begin{aligned}\mathcal{H} = J \sum_{j=1}^{L} \sum_{\sigma} & \mathcal{P} \left(c_{j,\sigma}^\dagger c_{j+1,\sigma} + c_{j,\sigma} c_{j+1,\sigma}^\dagger \right) \mathcal{P} \\ & + 4J \sum_{j=1}^{L} \{ (\boldsymbol{S}_j \boldsymbol{S}_{j+1} + n_j n_{j+1}/4) (\boldsymbol{\tau}_j \boldsymbol{\tau}_{j+1} + n_j n_{j+1}/4) - n_j n_{j+1}/4 \} \\ & - \mu \sum_{j=1}^{L} n_j - h \sum_{j=1}^{L} \left(g_S S_j^z + g_\tau \tau_j^z \right) , \end{aligned} \tag{2.26}$$

where σ runs over all possible electron configurations. Like in the t-J model the first term deals with the hopping of electrons, while the second term controls the nearest-neighbour interaction of spins and pseudospins. The remaining terms implement the effects of a chemical potential and an external magnetic field. In order to show the equivalence to the graded five-state Uimin-Sutherland model, we use the basis

$$|1\rangle = |{\uparrow_S}{\uparrow_\tau}\rangle\,, \qquad |2\rangle = |{\uparrow_S}{\downarrow_\tau}\rangle\,, \qquad |3\rangle = |{\downarrow_S}{\uparrow_\tau}\rangle\,, \tag{2.27a}$$
$$|4\rangle = |{\downarrow_S}{\downarrow_\tau}\rangle\,, \qquad |5\rangle = |0\rangle\,, \tag{2.27b}$$

where the grading is $p(1) = p(2) = p(3) = p(4) = 0$ and $p(5) = 1$. Note that apart from the unoccupied state $|0\rangle$ this basis choice is the same as for the $SU(4)$ spin-orbital model. Now, the Hamiltonian (2.26) may be recast as

$$\mathcal{H} = \sum_{j=1}^{L} \pi_{j,j+1} + JL - (2J+\mu)\sum_{j=1}^{L} n_j - h\sum_{j=1}^{L}\left(g_S S_j^z + g_\tau \tau_j^z\right)\,. \tag{2.28}$$

Again, all terms that appear in addition to the permutation operators are diagonal in the given basis and can therefore be treated as external field terms via the general chemical potentials

$$\mu_1 = J + \mu + (g_S + g_\tau)h/2\,, \qquad \mu_2 = J + \mu + (g_S - g_\tau)h/2\,, \tag{2.29a}$$
$$\mu_3 = J + \mu - (g_S - g_\tau)h/2\,, \qquad \mu_4 = J + \mu - (g_S + g_\tau)h/2\,, \tag{2.29b}$$
$$\mu_5 = -J\,. \tag{2.29c}$$

We end up with a perfect equivalence of our particular spin-orbital model and the $sl(4|1)$-symmetric case of the Uimin-Sutherland model.

Finally, we like to note that an analogous application of the $sl(4|1)$-symmetric case in order to treat the generalization of the spin-ladder system from Section 2.2.5 with mobile defects is possible, but, unfortunately, does not make much sense, since empty sites would only be allowed to appear simultaneous at both legs of the ladder. A model dealing with single empty sites, similar to a two-leg t-J ladder system, would require a larger number of basis states and, although perhaps possible to realize with some higher-rank Uimin-Sutherland model, is beyond the scope of the current work.

2.3 Quantum transfer matrix approach

In the following, we will be interested in obtaining the thermodynamics of the Uimin-Sutherland model in the thermodynamic limit (chain length $L \to \infty$). The basic problem is that the dimension of the Hilbert space increases exponentially in this limit, making a direct evaluation of the partition function $Z = \mathrm{Tr}\,\mathrm{e}^{-\beta\mathcal{H}}$ impossible. Fortunately, there exists an alternative way to obtain the partition function in terms of just the largest eigenvalue of the so-called quantum transfer matrix [40, 80]. In the following, we will briefly review the origin of this matrix and show that it can be diagonalized with the help of the Bethe ansatz.

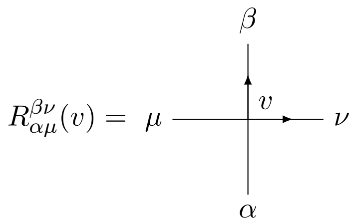


Figure 2.1: Graphical depiction of the R matrix as defined in equation (2.30).

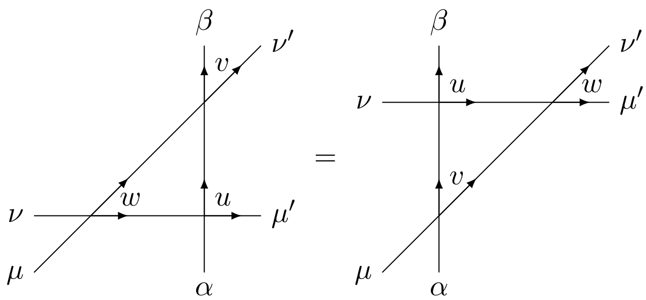


Figure 2.2: Graphical depiction of the Yang-Baxter equation, where $w = u - v$. Note that closed bonds between vertices denote summation over the intermediate variables.

2.3.1 The Perk-Schultz model

Let us first take a closer look at the Perk-Schultz model [65, 69], which turns out to be the classical two-dimensional counterpart of the one-dimensional Uimin-Sutherland model. The model can be defined as follows. Instead of a chain of length L, we now consider a square lattice with $L \times N$ vertices at the intersections, where a variable α, which can be in one of q many $\mathbb{Z}_2$-graded states ($\alpha = 1, \ldots, q$), is assigned to each bond of the lattice. Periodic boundary conditions are imposed in both directions, thus effectively contracting the lattice to a torus. Let us further associate the classical Boltzmann weight

$$R^{\beta\nu}_{\alpha\mu}(v) = \delta^{\nu}_{\alpha}\delta^{\beta}_{\mu} + v \cdot (-1)^{p(\alpha)p(\mu)} \cdot \delta^{\beta}_{\alpha}\delta^{\nu}_{\mu} \tag{2.30}$$

to every local vertex configuration $(\alpha, \beta, \mu, \nu)$, see Figure 2.1. Note that the free parameter v is the so-called spectral parameter. The total Boltzmann weight of some fixed lattice configuration is then given by the product of all local Boltzmann weights. Finally, we define the partition function of the Perk-Schultz model to be the sum of the Boltzmann weights over all possible lattice configurations.

The R matrix of Boltzmann weights (2.30) has some nice properties. First of all, it is $sl(r|s)$ symmetric, where r and s are again the numbers of states carrying the grading $p(\alpha) = 0$ and $p(\alpha) = 1$, respectively. More important, however, is the fact that it is a solution of the Yang-Baxter equation

$$\sum_{\rho,\sigma,\tau} R^{\rho\sigma}_{\mu\nu}(u-v) R^{\tau\mu'}_{\alpha\sigma}(u) R^{\beta\nu'}_{\tau\rho}(v) = \sum_{\rho,\sigma,\tau} R^{\tau\rho}_{\alpha\mu}(v) R^{\beta\sigma}_{\tau\nu}(u) R^{\nu'\mu'}_{\rho\sigma}(u-v)\,, \tag{2.31}$$

see Figure 2.2 on the preceding page for a graphical representation. Let us consider the monodromy matrix $T^{\nu'}_{\nu}(v)$ which is defined to be the product of all R matrices belonging to one row of the lattice,

$$T^{\nu'}_{\nu}(v) = \sum_{\{\mu\}} R^{\beta_1\mu_2}_{\alpha_1\nu}(v) R^{\beta_2\mu_3}_{\alpha_2\mu_2}(v) \cdots R^{\beta_{L-1}\mu_L}_{\alpha_{L-1}\mu_{L-1}}(v) R^{\beta_N\nu'}_{\alpha_L\mu_L}(v)\,, \tag{2.32}$$

where $\{\mu\}$ denotes the summation over all intermediate indices μ_j and the sets of indices α_j and β_j are suppressed. As a consequence of the Yang-Baxter equation (2.31), we see that the monodromy matrix is a representation of the corresponding Yang-Baxter algebra,

$$\sum_{\rho,\sigma} R^{\rho\sigma}_{\mu\nu}(u-v)\Big(T^{\mu'}_{\sigma}(u) T^{\nu'}_{\rho}(v)\Big) = \sum_{\rho,\sigma}\Big(T^{\rho}_{\mu}(v) T^{\sigma}_{\nu}(u)\Big) R^{\nu'\mu'}_{\rho\sigma}(u-v)\,. \tag{2.33}$$

Note that $(T^{\mu'}_{\mu} T^{\nu'}_{\nu})$ denotes a multiplication with respect to the suppressed space coordinates. Equation (2.33) may easily be proven by repeatedly applying the Yang-Baxter equation (2.31).

Let us further define the row-to-row transfer matrix of the Perk-Schultz model by taking the trace of the monodromy matrix,

$$\mathcal{T}(v) = \operatorname{Tr} \mathbf{T}(v) = \sum_{\nu} T^{\nu}_{\nu}(v)\,. \tag{2.34}$$

Admitting the multiple indices $\alpha = (\alpha_1 \ldots \alpha_L)$ and $\beta = (\beta_1 \ldots \beta_L)$ again, we find the explicit form

$$\mathcal{T}^{\beta}_{\alpha}(v) = \sum_{\{\nu\}} \prod_{j=1}^{L} R^{\beta_j \nu_{j+1}}_{\alpha_j \nu_j}(v)\,, \tag{2.35}$$

where $\nu_{L+1} = \nu_1$. The fact that the monodromy matrix is a representation of the Yang-Baxter algebra has an important consequence for the transfer matrix. To see this, let us multiply both sides of equation (2.33) with the inverse of $R^{\beta\nu}_{\alpha\mu}(u-v)$ and then take the trace with respect to the indices μ, μ' and ν, ν'. Due to the cyclic invariance of the trace, we immediately get the result that the transfer matrices constitute a commuting family, where

$$[\mathcal{T}(u), \mathcal{T}(v)] = 0 \tag{2.36}$$

for all $u, v \in \mathbb{C}$, which, in combination with the knowledge of the reference eigenstate

$$|\Omega\rangle = |1\,1\,\ldots\,1\rangle\,, \tag{2.37}$$

implies that the matrix may be diagonalized with the help of the nested Bethe ansatz, see [70, 74].

The usefulness of the transfer matrix stems from the fact that the partition function $Z_{\mathrm{PS}}(v)$ of the Perk-Schultz model can be written solely in terms of it,

$$Z_{\mathrm{PS}}(v) = \mathrm{Tr}\,\mathcal{T}^N(v)\,, \tag{2.38}$$

where the trace is meant to be taken in the q^L-dimensional space. Moreover, making use of Baxter's formula [10], one exactly recovers the Hamiltonian of the Uimin-Sutherland from the transfer matrix,

$$\mathcal{H}_0 = J\,\frac{\mathrm{d}}{\mathrm{d}v}\ln\mathcal{T}(v)\bigg|_{v=0} = J\sum_{j=1}^{L}\pi_{j,j+1}\,, \tag{2.39}$$

which gives the justification to view the Perk-Schultz model as the classical counterpart of the Uimin-Sutherland model. This connection also guarantees the integrability of the latter, since the eigenstates of the transfer matrix following from the Bethe ansatz are simultaneously eigenstates of the Hamiltonian $\mathcal{H}_0$.

2.3.2 Trotter-Suzuki mapping

Although equation (2.39) provides us with a means to calculate individual eigenvalues and eigenstates of the Hamiltonian $\mathcal{H}_0$ on the basis of the Bethe ansatz for the transfer matrix $\mathcal{T}(v)$, the evaluation of the partition function of the Uimin-Sutherland would still require the knowledge of all q^L many eigenvalues. Even then it would be difficult to deal with the thermodynamic limit $L \to \infty$. In the following, we therefore follow the approach which has been developed in [40, 80].

Let us introduce an alternative set of matrices $\overline{R}(v)$ which are obtained by rotating the graphical depiction of $R(v)$ counterclockwise by 90 degrees,

$$\overline{R}^{\beta\nu}_{\alpha\mu}(v) = R^{\mu\beta}_{\nu\alpha}(v)\,. \tag{2.40}$$

The transfer matrix $\overline{\mathcal{T}}(v)$ is then defined as the product of matrices $\overline{R}(v)$ in analogy to equation (2.35). It is easy to check that the new R matrix is again a solution of the Yang-Baxter equation (2.31), implying that the transfer matrices $\overline{\mathcal{T}}(v)$ also constitute a commuting family. Applying Baxter's formula to the transfer matrix $\overline{\mathcal{T}}(v)$ again leads to the Hamiltonian of the Uimin-Sutherland model,

$$\mathcal{H}_0 = J\,\frac{\mathrm{d}}{\mathrm{d}v}\ln\overline{\mathcal{T}}(v)\bigg|_{v=0} = J\sum_{j=1}^{L}\pi_{j,j+1}\,. \tag{2.41}$$

Note that the transfer matrices $\mathcal{T}(v)$ and $\overline{\mathcal{T}}(v)$ do not commute in general. At the spectral parameter $v = 0$, however, both of them reduce to simple shift operators. It is easy to check, that $\mathcal{T}(0)$ and $\overline{\mathcal{T}}(0)$ are the right and left shift operators ($\mathrm{e}^{\mathrm{i}P}$ and $\mathrm{e}^{-\mathrm{i}P}$), respectively. Combining this information with (2.39) and (2.41) we obtain the expansion

$$\ln(\mathcal{T}(v)\overline{\mathcal{T}}(v)) = \underbrace{\ln(\mathcal{T}(0)\overline{\mathcal{T}}(0))}_{=0} + 2\mathcal{H}_0/J\cdot v + \mathcal{O}(v^2)\,. \tag{2.42}$$

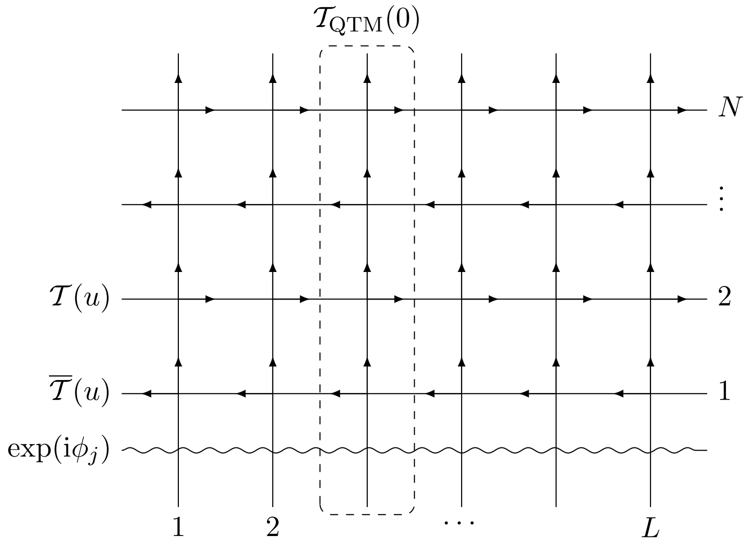


Figure 2.3: Graphical depiction of the staggered Perk-Schultz model. The transfer matrices $\mathcal{T}(v)$ and $\overline{\mathcal{T}}(v)$ alternate in vertical direction, while the wavy line indicates the twists at the boundaries. The quantum transfer matrix $\mathcal{T}_{\mathrm{QTM}}(v)$ is the inhomogeneous column-to-column transfer matrix (dashed box).

In order to relate this formula to the partition function of the Uimin-Sutherland model, we define $u = -J\beta/N$, where the Trotter number N is a sufficiently large even integer so that the absolute value of u is small compared to one. Then, we evaluate equation (2.42) at the spectral parameter $v = u$ and use the result to finally write down the partition function of the Uimin-Sutherland model Z in terms of the two transfer matrices,

$$Z = \operatorname{Tr} \mathrm{e}^{-\beta\mathcal{H}} = \lim_{N\to\infty} \operatorname{Tr}\left[\left(\mathcal{T}(u)\overline{\mathcal{T}}(u)\right)^{N/2} \mathrm{e}^{-\beta\mathcal{H}_{\mathrm{ext}}}\right], \tag{2.43}$$

where the trace has to be taken in the q^L-dimensional space.

Comparing this expression to equation (2.38) that we have obtained for the partition function of the Perk-Schultz model, we notice that Z may be viewed as the partition function of a staggered $L \times N$ Perk-Schultz model, see Figure 2.3. In this picture, we have alternating rows of matrices $R(u)$ and $\overline{R}(u)$ in the Trotter direction. The external field terms can be incorporated via twisted boundary conditions with imaginary twist angles $\phi_j = -\mathrm{i}\beta \sum_{\alpha=1}^{q} \mu_\alpha n_{j,\alpha}$, where j is the column number.

2.3.3 Definition of the quantum transfer matrix

Let us further exploit the interpretation of (2.43) being the partition function of a staggered Perk-Schultz vertex model. Recall that the two matrices $\mathcal{T}(u)$ and $\overline{\mathcal{T}}(u)$ act as row-to-row transfer matrices in this picture. As already indicated in Figure 2.3, the key idea is to consider the column-to-column transfer direction instead. Due to the fact that the R matrices alternate only in the vertical direction, we need just a single inhomogeneous transfer matrix here, which is called the quantum transfer matrix (QTM).

In order to properly define this matrix, let us first introduce the matrix $\widetilde{R}(v)$ which we get by rotating $R(v)$ clockwise by 90 degrees and changing the sign of the spectral parameter,

$$\widetilde{R}^{\beta\nu}_{\alpha\mu}(v) = R^{\nu\alpha}_{\mu\beta}(-v)\,. \tag{2.44}$$

By comparison with Figure 2.3 on the preceding page we then find the QTM to be

$$(\mathcal{T}_{\mathrm{QTM}})^{\beta}_{\alpha}(v) = \sum_{\{\nu\}} \mathrm{e}^{\beta\mu_{\nu_1}} \prod_{j=1}^{N/2} R^{\beta_{2j-1}\nu_{2j}}_{\alpha_{2j-1}\nu_{2j-1}}(\mathrm{i}v+u)\widetilde{R}^{\beta_{2j}\nu_{2j+1}}_{\alpha_{2j}\nu_{2j}}(\mathrm{i}v-u)\,, \tag{2.45}$$

where we have additionally introduced a spectral parameter v so that the QTMs constitute a commuting family, see Section 2.3.5 for the proof. Finally, we can write the partition function of the Uimin-Sutherland model solely in terms of the QTM,

$$Z = \lim_{N\to\infty} \mathrm{Tr}\,(\mathcal{T}_{\mathrm{QTM}}(0))^{L}\;. \tag{2.46}$$

2.3.4 Thermodynamic limit of the free energy

Let us now consider the thermodynamic limit ($L \to \infty$) of the free energy per unit length,

$$f = -\lim_{L\to\infty} \frac{1}{L\beta} \ln Z\,, \tag{2.47}$$

where Z is given by equation (2.46). The following facts will help to proceed:

1. The limits $L \to \infty$ and $N \to \infty$ may be interchanged, see [81].
2. The largest eigenvalue of the QTM is separated by a gap from the next-leading eigenvalues, see [77].

While the first statement allows us to take the limit $L \to \infty$ first and postpone the limit $N \to \infty$ to the very end of the calculation, it follows from the second one that the unique largest eigenvalue $\Lambda_{\max}(0)$ of the QTM dominates the trace for large L,

$$\begin{aligned} \ln \mathrm{Tr}\,(\mathcal{T}_{\mathrm{QTM}}(0))^{L} &= \ln\Bigg(\Lambda^{L}_{\max}(0) + \sum_{j=1}^{q^N-1} \Lambda^{L}_{j}(0)\Bigg) \\ &= L \ln \Lambda_{\max}(0) + \ln\left[1 + \sum_{j=1}^{q^N-1} \left(\frac{\Lambda_j(0)}{\Lambda_{\max}(0)}\right)^{L}\right], \end{aligned} \tag{2.48}$$

where the other eigenvalues are denoted by $\Lambda_j(0)$. Inserting the latter result into equation (2.47) and applying the thermodynamic limit finally yields

$$f = -\lim_{N\to\infty} \frac{1}{\beta} \ln \Lambda_{\max}(0)\,. \tag{2.49}$$

This is a remarkable result, since it tells us that the thermodynamical properties of the model are determined by just the unique largest eigenvalue of the QTM.

Note that the next-leading eigenvalues of the QTM also carry some information. They can be shown to determine the correlation lengths,

$$\xi_j = \left(\ln \frac{\Lambda_{\max}(0)}{\Lambda_j(0)} \right)^{-1} . \tag{2.50}$$

2.3.5 Commuting family of quantum transfer matrices

Like the homogeneous transfer matrices $\mathcal{T}(v)$ of the Perk-Schultz model, the QTMs $\mathcal{T}_{\mathrm{QTM}}(v)$ are constructed to constitute a commuting family, where

$$\left[\mathcal{T}_{\mathrm{QTM}}(v), \mathcal{T}_{\mathrm{QTM}}(v') \right] = 0 \tag{2.51}$$

for arbitrary $v, v' \in \mathbb{C}$. The reason is that both matrices $R(v)$ and $\widetilde{R}(v)$ which appear in the definition (2.45) are in fact representations of the same Yang-Baxter algebra generated by $R(v)$,

$$\sum_{\rho,\sigma,\tau} R^{\rho\sigma}_{\mu\nu}(v-v') R^{\tau\mu'}_{\alpha\sigma}(v) R^{\beta\nu'}_{\tau\rho}(v') = \sum_{\rho,\sigma,\tau} R^{\tau\rho}_{\alpha\mu}(v') R^{\beta\sigma}_{\tau\nu}(v) R^{\nu'\mu'}_{\rho\sigma}(v-v') , \tag{2.52a}$$

$$\sum_{\rho,\sigma,\tau} R^{\rho\sigma}_{\mu\nu}(v-v') \widetilde{R}^{\tau\mu'}_{\alpha\sigma}(v) \widetilde{R}^{\beta\nu'}_{\tau\rho}(v') = \sum_{\rho,\sigma,\tau} \widetilde{R}^{\tau\rho}_{\alpha\mu}(v') \widetilde{R}^{\beta\sigma}_{\tau\nu}(v) R^{\nu'\mu'}_{\rho\sigma}(v-v') . \tag{2.52b}$$

Then the quantum monodromy matrix defined by

$$\left(T_{\mathrm{QTM}}\right)^{\nu'}_{\nu}(v) = \sum_{\{\mu\}} R^{\beta_1\mu_2}_{\alpha_1\nu}(v-\mathrm{i}u) \widetilde{R}^{\beta_2\mu_3}_{\alpha_2\mu_2}(v+\mathrm{i}u) \cdots R^{\beta_{L-1}\mu_L}_{\alpha_{L-1}\mu_{L-1}}(v-\mathrm{i}u) \widetilde{R}^{\beta_N\nu'}_{\alpha_L\mu_L}(v+\mathrm{i}u) \tag{2.53}$$

will obviously be a representation as well,

$$\sum_{\rho,\sigma} R^{\rho\sigma}_{\mu\nu}(v-v') \Big(\left(T_{\mathrm{QTM}}\right)^{\mu'}_{\sigma}(v) \left(T_{\mathrm{QTM}}\right)^{\nu'}_{\rho}(v') \Big) = \sum_{\rho,\sigma} \Big(\left(T_{\mathrm{QTM}}\right)^{\rho}_{\mu}(v') \left(T_{\mathrm{QTM}}\right)^{\sigma}_{\nu}(v) \Big) R^{\nu'\mu'}_{\rho\sigma}(v-v') . \tag{2.54}$$

Since the QTM follows from the trace of the quantum monodromy matrix,

$$\mathcal{T}_{\mathrm{QTM}}(v) = \mathrm{Tr}\, \mathbf{T}_{\mathrm{QTM}}(\mathrm{i}v) = \sum_{\nu} \left(T_{\mathrm{QTM}}\right)^{\nu}_{\nu}(\mathrm{i}v) , \tag{2.55}$$

in analogy to equation (2.34), we can use the same arguments as in Section 2.3.1 to check that equation (2.51) is fulfilled.

2.3.6 Eigenvalues and Bethe ansatz equations

Since the quantum monodromy matrix (2.53) is a representation of the same Yang-Baxter algebra as the homogeneous monodromy matrix (2.32), the Bethe ansatz for both commuting sets of transfer matrices $\mathcal{T}_{\mathrm{QTM}}(v)$ and $\mathcal{T}(v)$ is very similar. The difference arises only from the different reference eigenstates. Note that in the case of the QTM a suitable reference state is given by

$$|\Omega\rangle = |1\,q\,\ldots\,1\,q\rangle\,. \tag{2.56}$$

The necessary modifications have been treated in [47]. Finally, the eigenvalues of the QTM are found to be

$$\Lambda(v) = \sum_{j=1}^{q} \lambda_j(v)\,, \tag{2.57}$$

where

$$\lambda_j(v) = \phi_-(v)\phi_+(v)\frac{q_{j-1}(v-\mathrm{i}\epsilon_j)}{q_{j-1}(v)}\frac{q_j(v+\mathrm{i}\epsilon_j)}{q_j(v)}\mathrm{e}^{\beta\mu_j}\,. \tag{2.58}$$

For convenience, we have defined $\epsilon_j = (-1)^{p(j)}$, $\phi_\pm(v) = (v \pm \mathrm{i}u)^{N/2}$ and

$$q_j(v) = \begin{cases} \phi_-(v) & \text{for } j = 0 \\ \prod_{k_j=1}^{M_j}(v - v^j_{k_j}) & \text{for } j = 1,\ldots,q-1 \\ \phi_+(v) & \text{for } j = q \end{cases}\,. \tag{2.59}$$

Note that we have used $u = -J\beta/N$ again. The complex parameters $v^j_{k_j}$, which are the roots of the polynomials $q_j(v)$, are called the Bethe ansatz roots. M_j gives the number of Bethe ansatz roots of the jth polynomial. Depending on the eigenvalue of interest, the numbers M_j can range from 0 to $N/2$.

The Bethe ansatz roots are not arbitrary, but have to fulfil a set of coupled nonlinear equations, the so-called Bethe ansatz equations. These arise either from the Bethe ansatz itself or, alternatively, from the requirement that the eigenvalues—like the QTM itself—are analytic in the spectral parameter. For each of the Bethe ansatz roots there is one Bethe ansatz equation,

$$\frac{\lambda_j(v^j_{k_j})}{\lambda_{j+1}(v^j_{k_j})} = -1\,, \tag{2.60}$$

to ensure that all potential poles vanish,

$$\mathrm{Res}_{v=v^j_{k_j}}\left(\lambda_j(v) + \lambda_{j+1}(v)\right) = 0\,. \tag{2.61}$$

From the form of (2.57) it then follows that all eigenvalues must in fact be polynomials of degree N. Note that the Bethe ansatz equations are solved not only by the Bethe ansatz roots, but that there exist additional solutions, the so-called hole solutions $v'^j_{l_j}$. From the structure of the functions $\lambda_j(v)$ it is clear, that there must exist a total of $M_{j-1} + M_{j+1} -$

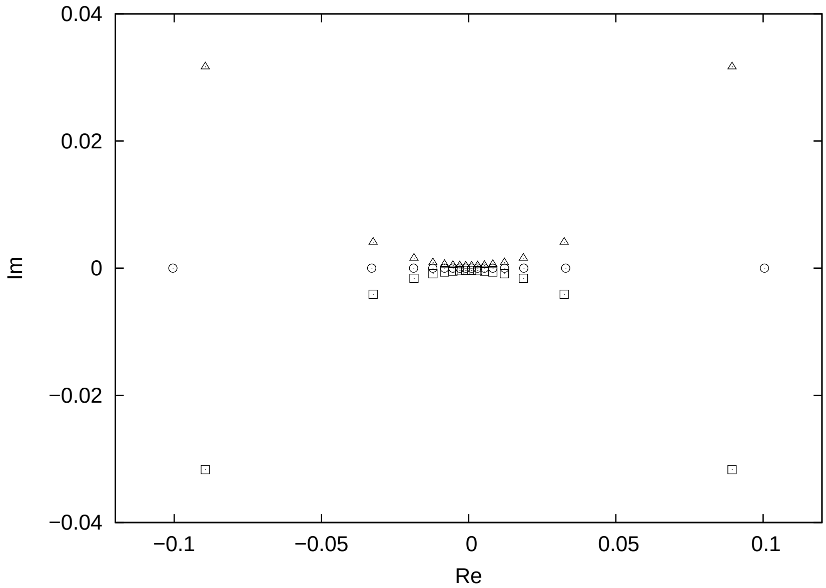


Figure 2.4: Distribution of the Bethe ansatz roots for $q = 4$, $N = 32$ and $J\beta = 0.32$. The three sets of roots are denoted by different symbols.

$\delta_{\epsilon_j,-\epsilon_{j+1}} M_j$ many hole solutions for the jth Bethe ansatz equation, where we have defined $M_0 = M_q = N/2$.

For the derivation of the thermodynamic properties we are only interested in the largest eigenvalue, for which the number of Bethe ansatz roots in each set is fixed to $M_j = N/2$. Figure 2.4 shows a typical distribution of the Bethe ansatz roots. Note that the distribution remains discrete even for large N, but that there is a cluster point at the origin. In order to perform the Trotter limit $N \to \infty$, it is thus not possible to introduce meaningful root densities. Note also that the distribution of hole solutions looks very similar. These are also lying on slightly curved lines with cluster points in the middle, but here the lines are shifted by $\pm$i away from the real axis.

We like to stress two further properties of the result for the eigenvalue (2.57). Firstly, for fixed total numbers r and s of basis states α with grading $p(\alpha) = 0$ and $p(\alpha) = 1$, respectively, the eigenvalue does not depend on the actual choice of grading. This fact, which is rather obvious from the definition of the Hamiltonian (2.2), but not obvious from equation (2.57), has certain consequences for the Bethe ansatz roots, see Appendix A for details. Secondly, the eigenvalue stays invariant if we change the sign of the coupling constant J and simultaneously reverse all gradings. For the following derivation of nonlinear integral equations, it is therefore sufficient to treat only cases where $r \geq s$.

Chapter 3

Fusion hierarchy and thermodynamic Bethe ansatz equations

Of course, it is in principle possible to numerically solve the Bethe ansatz equations (2.60) for some fixed Trotter number N. However, this approach is possible only for finite N and also quite cumbersome to perform. As we are interested in the limit $N \to \infty$ for deriving the free energy of the model, we have to choose a different approach, where we encode the Bethe ansatz equations into an alternative form for which the limit can be taken analytically.

One way to do so is the so-called thermodynamic Bethe ansatz (TBA), which has been developed by Yang & Yang, Takahashi and Gaudin [30, 82, 98, 99] for the case of the spin-1/2 Heisenberg chain using the string hyptothesis. Later it was realized by Jüttner, Klümper and Suzuki [38] that the TBA equations can also be derived by an alternative method. Instead of the string hypothesis, they exploit functional relations based on the fusion hierarchy of transfer matrices, which have been developed in a series of preceeding articles [11, 39, 46, 53, 88, 89].

Unfortunately, the TBA approach typically yields an infinite number of nonlinear integral equations (NLIEs), which strongly limits its practical use for numerical calculations. However, since the TBA equations can be calculated for arbitrary cases of the Uimin-Sutherland model in a quite straightforward way, and since they are closely related to the finite sets of nonlinear integral equations that we will derive in Chapter 4, it is worthwhile to explore their structure first.

We will start with a brief introduction of the fusion hierarchy relations. Then, we will use these relations to derive the TBA equations for the $sl(n)$- and $sl(n|1)$-symmetric cases of the Uimin-Sutherland model.

3.1 Fusion hierarchy relations

The fusion hierarchy relations for the transfer matrix eigenvalues of the general $sl(r|s)$-symmetric Perk-Schultz model have first been established in [88]. The necessary modifications for the QTM case, which basically shares the same structure, are given in [38]. Since, however, the underlying derivation is quite involved, we will content ourselves with just stating the results here.

3.2 The Yangian analogue of Young tableaux

In order to formulate the fusion relations, we will utilize a compact notation using the Yangian analogue of Young tableaux, which has been developed in [11, 54, 55, 75]. The simplest case is given by a box filled with the letter j,

$$\left.\boxed{j}\right|_v = \lambda_j(v)\,, \tag{3.1}$$

where the function $\lambda_j(v)$ is the one defined in equation (2.58). This corresponds to a Young tableau belonging to a vector of the first, $(r+s)$-dimensional representation of $sl(r|s)$. Similarly, expressions for general rectangular Young tableaux corresponding to higher dimensional representations are defined,

$$\left.\begin{array}{|c|c|c|}\hline n_{1,1} & \cdots & n_{1,m} \\ \hline \vdots & \ddots & \vdots \\ \hline n_{a,1} & \cdots & n_{a,m} \\ \hline \end{array}\right|_v = \prod_{j=1}^{a}\prod_{k=1}^{m} \left.\boxed{n_{j,k}}\right|_{v+\mathrm{i}(j-a/2)-\mathrm{i}(k-m/2)}\,, \tag{3.2}$$

where the shift in the spectral parameters on the right hand side depends on the coordinate of the corresponding box in the tableau, and the set of numbers $n_{j,k} \in \{1,\ldots,r+s\}$ has to fulfil two admissibility conditions, which depend on the grading:

1. If $p(n_{j,k}) = 0$, we demand $n_{j-1,k} < n_{j,k}$ and $n_{j,k-1} \leq n_{j,k}$.
2. If $p(n_{j,k}) = 1$, we demand $n_{j-1,k} \leq n_{j,k}$ and $n_{j,k-1} < n_{j,k}$.

In the $sl(r)$-symmetric case, for example, only semi-standard tableaux are allowed. This implements the combinatorial rules, which will be necessary in order to define suitable generalized eigenvalue functions in the next section.

3.3 Fused eigenvalue functions and functional relations

Let us define generalized fused eigenvalue functions as a sum over all admissible Young tableaux with certain height a and width m,

$$\Lambda_m^{(a)}(x) = \sum_{\{n_{j,k}\}^*} \left.\begin{array}{|c|c|c|}\hline n_{1,1} & \cdots & n_{1,m} \\ \hline \vdots & \ddots & \vdots \\ \hline n_{a,1} & \cdots & n_{a,m} \\ \hline \end{array}\right|_x\,, \tag{3.3}$$

where $\{n_{j,k}\}^*$ denotes all admissible sets of numbers. For those special cases, where there exist no admissible sets at all, we set

$$\Lambda_m^{(a)}(x) = 1 \qquad \text{if } a = 0 \text{ or } m = 0\,, \tag{3.4}$$

$$\Lambda_m^{(a)}(x) = 0 \qquad \text{if } a > r \text{ and } m > s\,. \tag{3.5}$$

It is important to note that all functions $\Lambda_m^{(a)}(x)$ are analytic as a direct consequence of the Bethe ansatz equations (2.60). Note also that the functions $\Lambda_m^{(a)}(x)$ are in general expected to be eigenvalues of corresponding generalized fused QTMs $\mathcal{T}_m^{(a)}(x)$, confer [38]. Since, however, the following calculations do not depend on this observation, we will not go into further details. Nevertheless, it is an important fact that the eigenvalues of the usual QTM (2.57) are contained as the simplest case,

$$\Lambda(v) = \Lambda_1^{(1)}(v) = \sum_{j=1}^{r+s} \boxed{j}\big|_v\,, \tag{3.6}$$

which is rather obvious from the definition.

Having all ingredients at hand, we are now ready to formulate the fusion hierarchy relations amongst the generalized eigenvalue functions,

$$\Lambda_m^{(a)}(x - \mathrm{i}/2)\Lambda_m^{(a)}(x + \mathrm{i}/2) = \Lambda_m^{(a-1)}(x)\Lambda_m^{(a+1)}(x) + \Lambda_{m-1}^{(a)}(x)\Lambda_{m+1}^{(a)}(x)\,. \tag{3.7}$$

They can be proved using a quantum analogue of the Jacobi-Trudi and Giambelli formulae, see [88, 89].

There is one further refinement. In order to actually work with the relations, it is possible and advantageous to divide all eigenvalue functions $\Lambda_m^{(a)}(x)$ by common factors, so that the resulting normalized functions $\widetilde{\Lambda}_m^{(a)}(x)$ are all just polynomials of degree N,

$$\Lambda_m^{(a)}(x) = n_m^{(a)}(x)\widetilde{\Lambda}_m^{(a)}(x)\,. \tag{3.8}$$

The corresponding normalization function $n_m^{(a)}(x)$ is found to be

$$n_m^{(a)}(x) = \prod_{j=1}^{a-1}\left\{\phi_-\left(x + \frac{a+m-2j}{2}\mathrm{i}\right)\phi_+\left(x - \frac{a+m-2j}{2}\mathrm{i}\right)\right\} \times \prod_{j=1}^{a} h_m^{(1)}\left(x - \frac{a+1-2j}{2}\mathrm{i}\right)\,, \tag{3.9a}$$

$$h_m^{(1)}(x) = \prod_{k=1}^{m-1}\left\{\phi_-\left(x - \frac{1+m-2k}{2}\mathrm{i}\right)\phi_+\left(x + \frac{1+m-2k}{2}\mathrm{i}\right)\right\}\,, \tag{3.9b}$$

with the exception of two special cases,

$$n_m^{(0)}(x) = \frac{1}{\phi_-(x + \frac{m}{2}\mathrm{i})\phi_+(x - \frac{m}{2}\mathrm{i})}\,, \qquad n_0^{(a)}(x) = \frac{1}{\phi_-(x - \frac{a}{2}\mathrm{i})\phi_+(x + \frac{a}{2}\mathrm{i})}\,. \tag{3.10}$$

Because of the identity

$$n_m^{(a)}(x-\mathrm{i}/2)n_m^{(a)}(x+\mathrm{i}/2) = n_m^{(a-1)}(x)n_m^{(a+1)}(x) = n_{m-1}^{(a)}(x)n_{m+1}^{(a)}(x)\,, \tag{3.11}$$

the fusion relations for the new set of functions $\widetilde{\Lambda}_m^{(a)}(x)$ even remain unchanged. Their final form is given by

$$\widetilde{\Lambda}_m^{(a)}(x-\mathrm{i}/2)\widetilde{\Lambda}_m^{(a)}(x+\mathrm{i}/2) = \widetilde{\Lambda}_m^{(a-1)}(x)\widetilde{\Lambda}_m^{(a+1)}(x) + \widetilde{\Lambda}_{m-1}^{(a)}(x)\widetilde{\Lambda}_{m+1}^{(a)}(x)\,. \tag{3.12}$$

3.4 Thermodynamic Bethe ansatz equations

In the following, we will focus only on the TBA equations for the largest eigenvalue of the QTM, $\Lambda_1^{(1)}(x) = \Lambda_{\max}(x)$, although TBA equations for other eigenvalues may also be obtained [38]. It is a remarkable fact that, although we are only interested in a single eigenvalue, the complete set of fusion relations has to be exploited.

In addition to the fusion relations (3.12), we will also need some knowledge on the analyticity properties of the fused eigenvalue functions. The correpsonding data are known from numerical case studies at finite Trotter number N, where the roots of the polynomials $\widetilde{\Lambda}_m^{(a)}(x)$ have been obtained by directly solving the Bethe ansatz equations for various models and parameters a and m. Based on these results, a general pattern can be conjectured. For each function $\widetilde{\Lambda}_m^{(a)}(x)$, we expect that half the roots are located above and below the real axis on slightly curved lines with imaginary parts close to $\pm(a+m)/2$.

Note that, in the following, we will only deal with the TBA equations for the $sl(n)$- and $sl(n|1)$-symmetric cases of the Uimin-Sutherland model and do not attempt to derive the TBA equations for the most general $sl(r|s)$-symmetric case. This is, because the general derivation—though structurally similar to the special cases—is more involved, and because we do not need these results in the following chapters. Nevertheless, the corresponding results already exist in the literature. They have been derived by Saleur [66] on the basis of the string-hypothesis.

3.4.1 Definiton of the auxiliary functions

In order to derive the thermodynamic Bethe ansatz equations from the fusion relations (3.12) we define the auxiliary functions

$$y_m^{(a)}(x) = \frac{\widetilde{\Lambda}_{m-1}^{(a)}(x)\widetilde{\Lambda}_{m+1}^{(a)}(x)}{\widetilde{\Lambda}_m^{(a-1)}(x)\widetilde{\Lambda}_m^{(a+1)}(x)}\,. \tag{3.13}$$

Additionally, we will also use the functions

$$Y_m^{(a)}(x) = y_m^{(a)}(x) + 1\,, \qquad \overline{Y}_m^{(a)}(x) = \left(y_m^{(a)}(x)\right)^{-1} + 1 = \frac{Y_m^{(a)}(x)}{y_m^{(a)}(x)}\,. \tag{3.14}$$

Due to the fusion relations (3.7) the auxiliary functions of the latter type can also be identified as rational functions in terms of $\widetilde{\Lambda}_m^{(a)}(x)$,

$$Y_m^{(a)}(x) = \frac{\widetilde{\Lambda}_m^{(a)}(x-\mathrm{i}/2)\widetilde{\Lambda}_m^{(a)}(x+\mathrm{i}/2)}{\widetilde{\Lambda}_m^{(a-1)}(x)\widetilde{\Lambda}_m^{(a+1)}(x)}\,, \qquad \overline{Y}_m^{(a)}(x) = \frac{\widetilde{\Lambda}_m^{(a)}(x-\mathrm{i}/2)\widetilde{\Lambda}_m^{(a)}(x+\mathrm{i}/2)}{\widetilde{\Lambda}_{m-1}^{(a)}(x)\widetilde{\Lambda}_{m+1}^{(a)}(x)}\,. \tag{3.15}$$

Concerning the analyticity properties of the auxiliary functions $y_m^{(a)}(x)$, $Y_m^{(a)}(x)$ and $\overline{Y}_m^{(a)}(x)$, we like to note that, with the exception of the case $a = m = 1$, they are all analytic, non-zero and have constant asymptotics (ANZC) inside the strip $-1/2 < \Im(x) < 1/2$, which directly follows from the root distribution of the eigenvalues $\widetilde{\Lambda}_m^{(a)}(x)$.

3.4.2 TBA equations for the $sl(n)$-symmetric case

The functions $Y_0^{(a)}(x) = \overline{Y}_m^{(0)}(x) = \overline{Y}_m^{(n)}(x) = 1$ are trivial in the $sl(n)$ case. For the other functions we find the relations

$$y_m^{(a)}(x-\mathrm{i}/2)y_m^{(a)}(x+\mathrm{i}/2) = \frac{Y_{m-1}^{(a)}(x)Y_{m+1}^{(a)}(x)}{\overline{Y}_m^{(a-1)}(x)\overline{Y}_m^{(a+1)}(x)}\,, \tag{3.16}$$

where $a = 1, \ldots, n-1$, as a direct consequence of the fusion relations (3.12). Furthermore, we find that

$$\widetilde{\Lambda}_1^{(n)}(x) = \phi_-\left(x - \frac{n+1}{2}\mathrm{i}\right)\phi_+\left(x + \frac{n+1}{2}\mathrm{i}\right)\mathrm{e}^{\beta\sum_{j=1}^n \mu_j}\,. \tag{3.17}$$

In order to decouple the set of functional relations we apply a Fourier transform to the logarithmic derivative of both sides of (3.16),

$$\widehat{f}(k) = \int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}\left[\ln f(x)\right]\mathrm{e}^{-\mathrm{i}kx}\,\frac{\mathrm{d}x}{2\pi}\,, \tag{3.18}$$

which directly yields

$$\begin{aligned}(\mathrm{e}^{-k/2}+\mathrm{e}^{k/2})\widehat{y}_m^{(a)}(k) = {}& \delta_{a,1}\delta_{m,1}\widehat{\phi}_h(k) + (1-\delta_{m,1})\widehat{Y}_{m-1}^{(a)}(k) + \widehat{Y}_{m+1}^{(a)}(k) \\ & - (1-\delta_{a,1})\widehat{\overline{Y}}_m^{(a-1)}(k) - (1-\delta_{a,n-1})\widehat{\overline{Y}}_m^{(a+1)}(k)\,.\end{aligned} \tag{3.19}$$

The extra term for $a = 1$ and $m = 1$,

$$\widehat{\phi}_h(k) = \begin{cases}\widehat{\phi}_-(k) - \widehat{\phi}_+(k) & \text{for } k < 0 \\ \widehat{\phi}_+(k) - \widehat{\phi}_-(k) & \text{for } k > 0\end{cases}\,, \tag{3.20}$$

stems from two explicitly known roots and poles of the function $y_1^{(1)}(x)$ that are located in the strip $-1/2 < \Im(x) < 1/2$. Equation (3.19) may be recast as

$$\begin{aligned}\sum_{b=1}^{n-1} M^{(a,b)}(k)\widehat{y}_m^{(b)}(k) = {}& \delta_{a,1}\delta_{m,1}\widehat{\phi}_h(k) + (1-\delta_{m,1})\widehat{Y}_{m-1}^{(a)}(k) + \widehat{Y}_{m+1}^{(a)}(k) \\ & - (1-\delta_{a,1})\widehat{Y}_m^{(a-1)}(k) - (1-\delta_{a,n-1})\widehat{Y}_m^{(a+1)}(k)\,,\end{aligned} \tag{3.21}$$

where we have defined the matrix $\mathbf{M}(k)$ with elements

$$M^{(a,b)}(k) = 2\cosh(k/2)\delta_{a,b} - \delta_{a,b-1} - \delta_{a,b+1}\,. \tag{3.22}$$

The inverse of this matrix is given by

$$\widehat{A}^{(a,b)}_{[n]}(k) = \frac{\sinh(\min(a,b)k/2)\sinh([n-\max(a,b)]k/2)}{\sinh(k/2)\sinh(nk/2)}\,. \tag{3.23}$$

Applying this information to (3.21) leads to

$$\begin{aligned}\widehat{y}^{(a)}_m(k) = \delta_{m,1}\widehat{A}^{(a,1)}_{[n]}(k)\widehat{\phi}_h(k) + \sum_{b=1}^{n-1}\Big\{\widehat{A}^{(a,b)}_{[n]}(k)\left((1-\delta_{m,1})\widehat{Y}^{(b)}_{m-1}(k) + \widehat{Y}^{(b)}_{m+1}(k)\right)\\ - \left(\widehat{A}^{(a,b-1)}_{[n]}(k) + \widehat{A}^{(a,b+1)}_{[n]}(k)\right)\widehat{Y}^{(b)}_m(k)\Big\}\,,\end{aligned} \tag{3.24}$$

which, after applying the Trotter limit $N \to \infty$, is finally transformed back to

$$\begin{aligned}\ln y^{(a)}_m(x) = -J\beta\delta_{m,1}A^{(a,1)}_{[n]}(x) + \sum_{b=1}^{n-1}\Big\{\Big[A^{(a,b)}_{[n]} * \left((1-\delta_{m,1})\ln Y^{(b)}_{m-1} + \ln Y^{(b)}_{m+1}\right)\Big](x)\\ - \Big[\big(A^{(a,b-1)}_{[n]} + A^{(a,b+1)}_{[n]}\big) * \ln Y^{(b)}_m\Big](x)\Big\}\,,\end{aligned} \tag{3.25}$$

where

$$A^{(a,b)}_{[n]}(x) = \int_{-\infty}^{\infty}\widehat{A}^{(a,b)}_{[n]}(k)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k\,, \tag{3.26}$$

and convolutions are denoted by

$$\big[f * g\big](x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,\frac{\mathrm{d}y}{2\pi}\,. \tag{3.27}$$

The set of equations given by (3.25) for $a = 1,\dots,n-1$ and $m = 1,\dots,\infty$ are the so-called TBA equations. Together with the known asymptotics of the involved functions they provide an infinite set of coupled nonlinear integral equations, which uniquely determine all auxiliary functions $y^{(a)}_m(x)$.

Finally, the largest eigenvalue of the QTM $\Lambda^{(1)}_1(0)$ can be reconstructed from the set of auxiliary functions. In order to derive the corresponding equation, let us consider the expression for the functions $Y^{(a)}_m(x)$ given in (3.15) and take the Fourier transform of its logarithmic derivative, which can be written as

$$\widehat{Y}^{(a)}_m(k) + \delta_{a,1}\widehat{\Lambda}^{(0)}_m(k) + \delta_{a,n-1}\widehat{\Lambda}^{(n)}_m(k) = \sum_{b=1}^{n-1} M^{(a,b)}(k)\widehat{\Lambda}^{(b)}_m(k)\,. \tag{3.28}$$

Using the inverse of the matrix $\mathbf{M}(k)$ again, this can be recast as

$$\widehat{\Lambda}^{(a)}_m(k) = \widehat{A}^{(a,1)}_{[n]}(k)\widehat{\Lambda}^{(0)}_m(k) + \widehat{A}^{(a,n-1)}_{[n]}(k)\widehat{\Lambda}^{(n)}_m(k) + \sum_{b=1}^{n-1}\widehat{A}^{(a,b)}_{[n]}(k)\widehat{Y}^{(b)}_m(k)\,. \tag{3.29}$$

In order to apply this result to $\Lambda_1^{(1)}(0)$, we define the additional function

$$\underline{\Lambda}_1^{(1)}(x) = \frac{\Lambda_1^{(1)}(x)}{\phi_-(x-\mathrm{i})\phi_+(x+\mathrm{i})}\,, \tag{3.30}$$

which has the advantage of having constant asymptotics. In the Trotter limit $N \to \infty$ this definition simply yields $\ln \Lambda_1^{(1)}(0) = \ln \underline{\Lambda}_1^{(1)}(0) - J\beta$. From equation (3.29) it follows that

$$\widehat{\underline{\Lambda}}_1^{(1)}(k) = -\mathrm{i}N \sinh(kJ\beta/N)\mathrm{e}^{-|k|/2}\frac{\sinh([n-1]k/2)}{\sinh(nk/2)} + \sum_{a=1}^{n-1} \widehat{A}_{[n]}^{(1,a)}(k)\widehat{Y}_1^{(a)}(k)\,. \tag{3.31}$$

Only the first term explicitly depends on the Trotter number N, so that we just have to consider

$$\lim_{N\to\infty} N\sinh(kJ\beta/N) = kJ\beta \tag{3.32}$$

to apply the global Trotter limit $N \to \infty$. Transforming the result back, we arrive at

$$\ln \Lambda_1^{(1)}(0) = -\beta\left\{J\left[1-\frac{2}{n}\left(\psi(1)-\psi\left(\frac{1}{n}\right)\right)\right] - \frac{1}{n}\sum_{j=1}^{n}\mu_j\right\} + \sum_{a=1}^{n-1}\left[A_{[n]}^{(1,a)} * \ln Y_1^{(a)}\right](0)\,, \tag{3.33}$$

where $\psi(x)$ denotes the digamma function.

3.4.3 TBA equations for the $sl(n|1)$-symmetric case

The derivation of the TBA equations for the $sl(n|1)$-symmetric case of the Uimin-Sutherland model is very similar to the one for the $sl(n)$-symmetric case and therefore deferred to Appendix B. The difference is that we have one additional identity for the eigenvalue functions,

$$\widetilde{\Lambda}_m^{(n)}(x)/a_m^{(n)} = \widetilde{\Lambda}_1^{(n+m-1)}(x)/a_1^{(n+m-1)}\,, \tag{3.34}$$

where $a_m^{(b)}$ are the highest coefficients of the polynomials, and that only the functions $Y_0^{(a)}(x) = \overline{Y}_m^{(0)}(x) = 1$ are trivial; the function

$$y_1^{(n)}(x) = \frac{\widetilde{\Lambda}_0^{(n)}(x)\widetilde{\Lambda}_2^{(n)}(x)}{\widetilde{\Lambda}_1^{(n-1)}(x)\widetilde{\Lambda}_1^{(n+1)}(x)} = \frac{\widetilde{\Lambda}_0^{(n)}(x)a_2^{(n)}}{\widetilde{\Lambda}_1^{(n-1)}(x)a_1^{(n+1)}} \tag{3.35}$$

now contributes to the final equations.

Finally, we get the result

$$\begin{aligned}\ln y_1^{(a)}(x) = {}& -\beta\left(J\frac{4a}{4x^2+a^2} + c^{(a)}\right) + \sum_{b=1}^{n-1}\left[A_{[n]}^{(a,b)} * \ln Y_2^{(b)}\right](x) \\ & - \sum_{b=1}^{n}\left[\left(A_{[n]}^{(a,b-1)} + A_{[n]}^{(a,b+1)} + C^{(a,b)}\right) * \ln Y_1^{(b)}\right](x)\,,\end{aligned} \tag{3.36a}$$

$$\ln y_1^{(n)}(x) = -\beta\left(J\frac{4n}{4x^2+n^2} + c^{(n)}\right) - \sum_{b=1}^{n}\left[C^{(n,b)} * \ln Y_1^{(b)}\right](x) + \ln Y_1^{(n)}(x)\,, \tag{3.36b}$$

where $a = 1, \ldots, n-1$, and

$$\begin{aligned}\ln y_m^{(a)}(x) = \sum_{b=1}^{n-1}\Big\{&\left[A_{[n]}^{(a,b)} * \left(\ln Y_{m-1}^{(b)} + \ln Y_{m+1}^{(b)}\right)\right](x) \\ &- \left[\left(A_{[n]}^{(a,b-1)} + A_{[n]}^{(a,b+1)}\right) * \ln Y_m^{(b)}\right](x)\Big\}\,,\end{aligned} \tag{3.37}$$

where $m = 2, \ldots, \infty$ and $a = 1, \ldots, n-1$. Considering the asymptotes of these equations, we find the constants to be

$$c^{(a)} = a\mu_g - \frac{a}{n}\sum_{\substack{j=1\\ j\neq g}}^{n}\mu_j\,, \tag{3.38}$$

where g is the label of the basis state with grading $p(g) = 1$. The kernel functions $C^{(a,b)}(x)$ are defined by

$$C^{(a,b)}(x) = \int_{-\infty}^{\infty} \mathrm{e}^{-(n-1)|k|/2}\frac{\sinh(ak/2)\sinh(bk/2)}{\sinh(k/2)\sinh(nk/2)}\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k\,. \tag{3.39}$$

Again, the set of TBA equations (3.36) and (3.37) together with the known asymptotics completely determines all auxiliary functions.

Once the auxiliary functions are known, the largest eigenvalue of the QTM can be calculated via

$$\ln\Lambda_1^{(1)}(0) = \beta(J+\mu_g) + \sum_{a=1}^{n}\left[\frac{4a}{4x^2+a^2} * \ln Y_1^{(a)}\right](0)\,. \tag{3.40}$$

3.5 Zero-temperature limit

We have already seen that the TBA equations are a coupled set of infinitely many nonlinear integral equations. In the zero-temperature limit, nevertheless, they reduce to only a finite set of coupled linear integral equations. In order to take the limit, we first have to rescale the auxiliary functions. We define

$$e_m^{(a)}(x) = \frac{1}{\beta}\ln y_m^{(a)}(x)\,, \qquad E_m^{(a)}(x) = \frac{1}{\beta}\ln Y_m^{(a)}(x)\,. \tag{3.41}$$

Note that in the zero-temperature limit, the relation between both sets of functions is largely simplified,

$$E_m^{(a)}(x) \to e^{+}{}_m^{(a)}(x) = \begin{cases} e_m^{(a)}(x) & \text{if } \Re(e_m^{(a)}(x)) > 0 \\ 0 & \text{if } \Re(e_m^{(a)}(x)) \leq 0 \end{cases}\,, \tag{3.42}$$

which is also the cause for the structural simplifcation of the NLIEs in this limit.

Applying the limit to the TBA equations of the $sl(n)$ case (3.25), we finally arrive at the coupled set of linearized integral equations

$$e_1^{(a)}(x) = -\left(JA^{(a,1)}(x) + c^{(a)}\right) - \sum_{b=1}^{n-1}\left[\mathcal{K}_{[n]}^{(a,b)} * e^{+}{}_1^{(b)}\right](x)\,, \tag{3.43}$$

where

$$\mathcal{K}_{[n]}^{(a,b)} = \int_{-\infty}^{\infty}\left\{\mathrm{e}^{|k|/2}\widehat{A}^{(a,b)}(k) - \delta_{a,b}\right\}\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k\,. \tag{3.44}$$

Note that the constants $c^{(a)}$ have not been fixed here. They depend on the proportions of the general chemical potentials and vanish, if all of them are zero. The equation for the largest eigenvalue (3.33) turns into

$$\frac{1}{\beta}\ln\Lambda_1^{(1)}(0) = -\left\{J\left[1 - \frac{2}{n}\left(\psi(1) - \psi\left(\frac{1}{n}\right)\right)\right] - \sum_{j=1}^{n}\mu_j\right\} + \sum_{a=1}^{n-1}\left[A^{(1,a)} * e^{+}{}_1^{(a)}\right](0)\,. \tag{3.45}$$

In case of the $sl(n|1)$-symmetric Uimin-Sutherland model, see equations (3.36) and (3.37), the linearized zero-temperature equations have the structure,

$$e_1^{(a)}(x) = -\left(J\frac{4a}{4x^2+a^2} + c^{(a)}\right) - \sum_{b=1}^{n}\left[\mathcal{K}^{(a,b)} * e^{+}{}_1^{(a)}\right](x)\,, \tag{3.46}$$

where the constants $c^{(a)}$ still have to be fixed, and the integration kernels are given by

$$\mathcal{K}^{(a,b)}(x) = \int_{-\infty}^{\infty}\left\{\mathrm{e}^{-(\max(a,b)-1)|k|/2}\frac{\sinh(\min(a,b)k/2)}{\sinh(k/2)} - \delta_{a,b}\right\}\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k\,. \tag{3.47}$$

The corresponding modification of the equation for the largest eigenvalue (3.40) yields

$$\frac{1}{\beta}\ln\Lambda_1^{(1)}(0) = (J + \mu_g) + \sum_{a=1}^{n}\left[\frac{4a}{4x^2+a^2} * e^{+}{}_1^{(a)}\right](0)\,. \tag{3.48}$$

3.6 Concluding remarks

The advantage of the TBA approach lies in the fact that the corresponding equations can be obtained in a rather straightforward way for general $sl(r|s)$-symmetric Uimin-Sutherland models. However, they are less suitable for numerical investigations at finite temperature and external fields, since one has to deal with an infinite number of auxiliary functions. Some kind of truncation scheme is necessary. The standard method is to take a large but finite number of the auxiliary functions $y_m^{(a)}(x)$ and approximate the rest by their asymptotic values. Estimates for the systematic error introduced by this procedure, which have been obtained in comparison with direct Bethe ansatz solutions at finite Trotter numbers N

in [38], nevertheless show that it is difficult to obtain reliable numerical results in the low-temperature region in this way.

In order to get accurate numerical results, we have to use a different approach to obtain only finite numbers of coupled NLIEs, which will be presented in the next chapter. However, both approaches are mutually related, and the knowledge of the TBA equations therefore helps in order to understand the general structure of the alternative NLIEs.

Chapter 4

Finite sets of nonlinear integral equations

The TBA equations, which have been treated in the last chapter, can be obtained for the general $sl(r|s)$-symmetric Uimin-Sutherland model. Nevertheless, it is a major disadvantage of this approach that we have to deal with an infinite number of coupled nonlinear integral equations (NLIEs). Fortunately, there exist alternative approaches, where only a finite number of auxiliary functions is necessary.

One of them has been developed by Takahashi [84] for the case of the spin-1/2 Heisenberg chain, where only one single NLIE is sufficient. His approach was later generalized by Tsuboi [90, 91] to the case of the $sl(r|s)$-symmetric Uimin-Sutherland model. Here, the number of auxiliary functions and therefore the number of resulting coupled NLIEs is equal to the rank of the underlying algebra. However, integration contours along complex paths instead of simple convolutions—like in the TBA equations—are involved in the NLIEs, which makes numerical calculations difficult. Only high-temperature expansions, albeit to very high order (~ 40), have been obtained using this approach [71, 91, 92].

In the following we will stick to the third approach, which has been developed by Klümper, Batchelor and Pearce [40, 43, 44] and has been applied to the thermodynamics of the spin-1/2 Heisenberg chain by Klümper in [41]. There exist two equivalent formulations. The first one uses just a single auxiliary function, where the resulting NLIE involves complex integration contours. The second formulation uses a closely related additional auxiliary function and leads to two coupled NLIEs involving only convolution-type integrals. Due to its structure, this second form of the equations is well suited for numerical calculations at arbitrary finite temperature and chemical potentials, see Chapter 6.

Unfortunately, the extension of the Klümper-type approach to the arbitrary Uimin-Sutherland model is difficult since there is no known straightforward way to construct the required auxiliary functions. Therefore, the generalized auxiliary functions and NLIEs have previously been obtained only for cases with three components at most, $r + s \leq 3$, basically by trial and error.

Our treatment improves on the situation. We have succeeded in finding auxiliary functions for all cases with four components. We are also able to shed some light on the general structure of the NLIE and thus conjecture the NLIEs even for some higher-rank cases, namely for the $sl(5)$- and $sl(4|1)$-symmetric cases. A general construction, like in the case of the TBA or Takahashi-type equations, however, is still out of reach.

4.1 Previous work

In the following sections, we review the previous results on the NLIEs of Klümper-type. This is done in order to unify the notation and to introduce some additional observations on the structure of the integration kernels and the connection to the TBA approach, which will later help for treating the generalizations.

4.1.1 The $sl(2)$-symmetric case

Let us start with the $sl(2)$-symmetric case. Although this is the simplest non-trivial case, the basic ideas are the same as for the more complicated cases. We define two auxiliary functions [41],

$$b_{1,1}^{(1)}(x) = \left.\frac{\boxed{1}}{\boxed{2}}\right|_{x+\mathrm{i}/2}, \qquad b_{1,2}^{(1)}(x) = \left.\frac{\boxed{2}}{\boxed{1}}\right|_{x-\mathrm{i}/2}. \tag{4.1}$$

Additionally, we introduce the uppercase functions $B_{1,j}^{(1)}(x) = b_{1,j}^{(1)}(x)+1$. Note that the extra indices are not essential here. They are used solely in order to maintain a consistent notation which will be useful in the context of generalizations.

We note that the auxiliary functions $b_{1,j}^{(1)}(x)$ and $B_{1,j}^{(1)}(x)$ are rational functions in terms of the spectral parameter x. Moreover, it can be checked numerically that they are analytic, non-zero and have constant asymptotics (ANZC) in a strip $-1/2 \lesssim \Im(x) \lesssim 1/2$ surrounding the real axis. We recast the auxiliary functions in factorized form,

$$b_{1,1}^{(1)}(x) = \frac{\phi_-(x-\frac{\mathrm{i}}{2})\phi_+(x+\frac{\mathrm{i}}{2})q_1(x+\frac{3}{2}\mathrm{i})}{\phi_-(x+\frac{\mathrm{i}}{2})\phi_+(x+\frac{3}{2}\mathrm{i})q_1(x-\frac{\mathrm{i}}{2})} \cdot \mathrm{e}^{\beta(\mu_1-\mu_2)}, \tag{4.2a}$$

$$b_{1,2}^{(1)}(x) = \frac{\phi_-(x-\frac{\mathrm{i}}{2})\phi_+(x+\frac{\mathrm{i}}{2})q_1(x-\frac{3}{2}\mathrm{i})}{\phi_-(x-\frac{3}{2}\mathrm{i})\phi_+(x-\frac{\mathrm{i}}{2})q_1(x+\frac{\mathrm{i}}{2})} \cdot \mathrm{e}^{\beta(\mu_2-\mu_1)}, \tag{4.2b}$$

$$B_{1,1}^{(1)}(x) = \frac{q_1(x+\frac{\mathrm{i}}{2})\widetilde{\Lambda}_1^{(1)}(x+\frac{\mathrm{i}}{2})}{\phi_-(x+\frac{\mathrm{i}}{2})\phi_+(x+\frac{3}{2}\mathrm{i})q_1(x-\frac{\mathrm{i}}{2})} \cdot \frac{\mathrm{e}^{\beta\mu_1}+\mathrm{e}^{\beta\mu_2}}{\mathrm{e}^{\beta\mu_2}}, \tag{4.2c}$$

$$B_{1,2}^{(1)}(x) = \frac{q_1(x-\frac{\mathrm{i}}{2})\widetilde{\Lambda}_1^{(1)}(x-\frac{\mathrm{i}}{2})}{\phi_-(x-\frac{3}{2}\mathrm{i})\phi_+(x-\frac{\mathrm{i}}{2})q_1(x+\frac{\mathrm{i}}{2})} \cdot \frac{\mathrm{e}^{\beta\mu_1}+\mathrm{e}^{\beta\mu_2}}{\mathrm{e}^{\beta\mu_1}}, \tag{4.2d}$$

where the function $\widetilde{\Lambda}_1^{(1)}(x)$ is defined as the polynomial $\Lambda_1^{(1)}(x)$ divided by its highest coefficient,

$$\Lambda_1^{(1)}(x) = \widetilde{\Lambda}_1^{(1)}(x) \cdot (\mathrm{e}^{\beta\mu_1}+\mathrm{e}^{\beta\mu_2}). \tag{4.3}$$

Note that in the $sl(2)$-symmetric case the roots of the largest eigenvalue are equivalent to the hole solutions of the Bethe ansatz equation (2.60).

Due to the analyticity properties, we are allowed to apply a Fourier transform to the logarithmic derivative of all auxiliary functions,

$$\widehat{f}(k) = \int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} \left[\ln f(x)\right] \mathrm{e}^{-\mathrm{i}kx} \frac{\mathrm{d}x}{2\pi}\,, \tag{4.4}$$

in analogy to the derivation of the TBA equations in Chapter 3. In the cases $k<0$ and $k>0$, we will close the integration path to a contour above and below the real axis, respectively. Let us first turn to the case $k<0$, where we only have to deal with roots and poles, which are located above the real axis. Here, we get the result,

$$\widehat{b}^{(1)}_{1,1}(k) = \mathrm{e}^{k/2}\widehat{\phi}_-(k) - \mathrm{e}^{k/2}\widehat{q}_1(k)\,, \tag{4.5a}$$

$$\widehat{b}^{(1)}_{1,2}(k) = \mathrm{e}^{k/2}\widehat{\phi}_-(k) + \mathrm{e}^{3k/2}\widehat{q}_1(k) - \mathrm{e}^{3k/2}\widehat{\phi}_-(k) - \mathrm{e}^{k/2}\widehat{\phi}_+(k)\,, \tag{4.5b}$$

$$\widehat{B}^{(1)}_{1,1}(k) = \mathrm{e}^{-k/2}\widehat{\Lambda}^{(1)}_1(k) - \mathrm{e}^{k/2}\widehat{q}_1(k)\,, \tag{4.5c}$$

$$\widehat{B}^{(1)}_{1,2}(k) = \mathrm{e}^{k/2}\widehat{\Lambda}^{(1)}_1(k) + \mathrm{e}^{k/2}\widehat{q}_1(k) - \mathrm{e}^{3k/2}\widehat{\phi}_-(k) - \mathrm{e}^{k/2}\widehat{\phi}_+(k)\,. \tag{4.5d}$$

The latter two equations may easily be solved to get the functions $\widehat{q}_1(k)$ and $\widehat{\Lambda}^{(1)}_1(k)$ in terms of the auxiliary functions $\widehat{B}^{(1)}_{1,j}(k)$. We then substitute the result into the equations for the functions $\widehat{b}^{(1)}_{1,j}(k)$ and are left with a self-consistent set of equations for the auxiliary functions. After treating the case $k>0$ along the same lines, we combine the results to obtain equations valid for all $k \in \mathbb{R}$,

$$\widehat{b}^{(1)}_{1,1}(k) = -\frac{\mathrm{i}N\sinh(kJ\beta/N)}{2\cosh(k/2)} + \frac{\mathrm{e}^{-|k|/2}}{2\cosh(k/2)}\widehat{B}^{(1)}_{1,1}(k) - \frac{\mathrm{e}^{-k-|k|/2}}{2\cosh(k/2)}\widehat{B}^{(1)}_{1,2}(k)\,, \tag{4.6a}$$

$$\widehat{b}^{(1)}_{1,2}(k) = -\frac{\mathrm{i}N\sinh(kJ\beta/N)}{2\cosh(k/2)} - \frac{\mathrm{e}^{k-|k|/2}}{2\cosh(k/2)}\widehat{B}^{(1)}_{1,1}(k) + \frac{\mathrm{e}^{-|k|/2}}{2\cosh(k/2)}\widehat{B}^{(1)}_{1,2}(k)\,. \tag{4.6b}$$

Only the first term on the right hand side of each equation explicitly depends on the Trotter number N. Therefore, we can perform the global Trotter limit $N \to \infty$ just by replacing this term by its asymptotic form,

$$\lim_{N\to\infty} N\sinh(kJ\beta/N) = kJ\beta\,. \tag{4.7}$$

The remaining steps are to apply the inverse Fourier transform to equations (4.6) and integrate with respect to the spectral parameter x. This eventually yields

$$\ln b^{(1)}_{1,1}(x) = -\beta\left(J\frac{\pi}{\cosh(\pi x)} + c^{(1)}_1\right) - \left[K_0 * \ln B^{(1)}_{1,1}\right](x) - \left[K_1 * \ln B^{(1)}_{1,2}\right](x)\,, \tag{4.8a}$$

$$\ln b^{(1)}_{1,2}(x) = -\beta\left(J\frac{\pi}{\cosh(\pi x)} + c^{(1)}_2\right) - \left[K_2 * \ln B^{(1)}_{1,1}\right](x) - \left[K_0 * \ln B^{(1)}_{1,2}\right](x)\,, \tag{4.8b}$$

where convolutions are denoted by

$$\big[f * g\big](x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,\frac{\mathrm{d}y}{2\pi}, \tag{4.9}$$

and the kernel functions are given by

$$K_0(x) = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\frac{\Gamma(1+\frac{x}{2}\mathrm{i})\Gamma(\frac{1}{2}-\frac{x}{2}\mathrm{i})}{\Gamma(1-\frac{x}{2}\mathrm{i})\Gamma(\frac{1}{2}+\frac{x}{2}\mathrm{i})}\right), \tag{4.10a}$$

$$K_1(x) = K_0(x) + \frac{\mathrm{i}}{x+\mathrm{i}} - \frac{\mathrm{i}}{x-0\mathrm{i}}, \quad K_2(x) = K_0(x) + \frac{\mathrm{i}}{x+0\mathrm{i}} - \frac{\mathrm{i}}{x-\mathrm{i}}. \tag{4.10b}$$

Here, $x \pm 0\mathrm{i}$ indicates that the corresponding terms are strictly valid only for x with positive/negative imaginary part. The missing integration constants $c_j^{(1)}$ are determined by considering equation (4.8) in the limit $x \to \infty$. For the convolution terms, we find

$$\lim_{x\to\infty}\big[f * g\big](x) = g(\infty)\int_{-\infty}^{\infty} f(x)\,\frac{\mathrm{d}x}{2\pi}. \tag{4.11}$$

Since the asymptotic values of the auxiliary functions are known and the norms of the kernel functions are found to be $1/2$, the constants follow as

$$c_1^{(1)} = \frac{\mu_2-\mu_1}{2}, \qquad c_2^{(1)} = \frac{\mu_1-\mu_2}{2}. \tag{4.12}$$

Now, the finite set of coupled NLIEs (4.8) for the $sl(2)$-symmetric case is complete.

The largest eigenvalue of the QTM, however, has still to be recovered from the auxiliary functions. Recall that an expression for $\widehat{\Lambda}_1^{(1)}(k)$ in terms of the auxiliary functions $\widehat{B}_{1,j}^{(1)}(k)$ already appeared during the derivation of the NLIEs. In order to transform this expression back, we first introduce the additional function,

$$\underline{\Lambda}_1^{(1)}(x) = \frac{\Lambda_1^{(1)}(x)}{\phi_-(x-\mathrm{i})\phi_+(x+\mathrm{i})}, \tag{4.13}$$

which has a constant asymptotic value. This expression simplifies to $\ln\Lambda_1^{(1)}(0) = \underline{\Lambda}_1^{(1)}(0) - J\beta$ in the Trotter limit $N \to \infty$. We find

$$\widehat{\underline{\Lambda}}_1^{(1)}(k) = \frac{iN\sinh(kJ\beta/N)\mathrm{e}^{-|k|/2}}{2\cosh(k/2)} + \frac{1}{2\cosh(k/2)}\left(\widehat{B}_{1,1}^{(1)}(k) + \widehat{B}_{1,2}^{(1)}(k)\right). \tag{4.14}$$

Again, we take the Trotter limit $N \to \infty$ by means of equation (4.7). After the inverse transform, we fix the integration constants like in the derivation of the NLIEs. This finally yields

$$\ln\Lambda_1^{(1)}(0) = -\beta\left\{J(1-2\ln 2) - \frac{\mu_1+\mu_2}{2}\right\} + \left[\frac{\pi}{\cosh(\pi x)} * \left(\ln B_{1,1}^{(1)} + \ln B_{1,2}^{(1)}\right)\right](0). \tag{4.15}$$

The equation is basically the same as the one we derived for the eigenvalue in the TBA approach (3.33) for $n = 2$. This is expected, since the functions $Y_1^{(1)}(x)$ and $B_{1,j}^{(1)}(x)$ are related via

$$Y_1^{(1)}(x) = B_{1,1}^{(1)}(x)B_{1,2}^{(1)}(x). \tag{4.16}$$

4.1.2 The $sl(3)$-symmetric case

We have seen in the previous section that the definition of suitable auxiliary functions is the key to the derivation of a finite set of nonlinear integral equations. The necessary auxiliary functions for the $sl(3)$-symmetric case have first been found by Fujii and Klümper [27]. There are six auxiliary functions

$$b^{(1)}_{1,1}(x) = \left.\frac{\boxed{1}}{\boxed{2}+\boxed{3}}\right|_{x+\mathrm{i}/2}, \qquad b^{(1)}_{1,2}(x) = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\cdot\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right)}\right|_{x}, \tag{4.17a}$$

$$b^{(1)}_{1,3}(x) = \left.\frac{\boxed{3}}{\boxed{1}+\boxed{2}}\right|_{x-\mathrm{i}/2}, \qquad b^{(2)}_{1,1}(x) = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}}\right|_{x+\mathrm{i}/2}, \tag{4.17b}$$

$$b^{(2)}_{1,2}(x) = \left.\frac{\boxed{1}\cdot\boxed{3}}{\boxed{2}\cdot\left(\boxed{1}+\boxed{2}+\boxed{3}\right)}\right|_{x}, \qquad b^{(2)}_{1,3}(x) = \left.\frac{\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}}\right|_{x-\mathrm{i}/2}. \tag{4.17c}$$

Again, we define uppercase auxiliary functions via $B^{(a)}_{1,j}(x) = b^{(a)}_{1,j}(x) + 1$. Like in the $sl(2)$-symmetric case, there is a close connection to the auxiliary functions of the TBA approach,

$$Y^{(a)}_1(x) = B^{(a)}_{1,1}(x)B^{(a)}_{1,2}(x)B^{(a)}_{1,3}(x) \qquad (a = 1, 2)\,. \tag{4.18}$$

The auxiliary functions may apparently be grouped into two subsets, which justifies our naming convention. Each of the subsets contains three functions, which corresponds to the dimension of the two fundamental representations of $sl(3)$. Moreover, the second set of functions may be obtained from the first one by a conjugation transformation of the Young tableaux, confer [27].

Again, the auxiliary functions have the ANZC property in a strip $-1/2 \lesssim \Im(x) \lesssim 1/2$. The factorized form is

$$b^{(1)}_{1,1}(x) = \frac{\phi_-(x-\frac{\mathrm{i}}{2})\phi_+(x+\frac{\mathrm{i}}{2})q_1(x+\frac{3}{2}\mathrm{i})}{\phi_-(x+\frac{\mathrm{i}}{2})q^{(h)}_2(x+\frac{\mathrm{i}}{2})}\cdot\frac{\mathrm{e}^{\beta\mu_1}}{\mathrm{e}^{\beta\mu_2}+\mathrm{e}^{\beta\mu_3}}\,, \tag{4.19a}$$

$$b^{(1)}_{1,2}(x) = \frac{\phi_-(x-\frac{\mathrm{i}}{2})\phi_+(x+\frac{\mathrm{i}}{2})q_1(x-\frac{3}{2}\mathrm{i})q_2(x+\frac{3}{2}\mathrm{i})}{q_1(x+\frac{\mathrm{i}}{2})q_2(x-\frac{\mathrm{i}}{2})\widetilde{\Lambda}^{(2)}_1(x)}\cdot\frac{\mathrm{e}^{2\beta\mu_2}}{a^{(2)}_1}\,, \tag{4.19b}$$

$$b^{(1)}_{1,3}(x) = \frac{\phi_-(x-\frac{\mathrm{i}}{2})\phi_+(x+\frac{\mathrm{i}}{2})q_2(x-\frac{3}{2}\mathrm{i})}{\phi_+(x-\frac{\mathrm{i}}{2})q^{(h)}_1(x-\frac{\mathrm{i}}{2})}\cdot\frac{\mathrm{e}^{\beta\mu_3}}{\mathrm{e}^{\beta\mu_1}+\mathrm{e}^{\beta\mu_2}}\,, \tag{4.19c}$$

$$b^{(2)}_{1,1}(x) = \frac{\phi_-(x-\mathrm{i})\phi_+(x+\mathrm{i})q_2(x+2\mathrm{i})}{\phi_+(x+2\mathrm{i})q^{(h)}_1(x)}\cdot\frac{\mathrm{e}^{\beta(\mu_1+\mu_2)}}{\mathrm{e}^{\beta\mu_3}(\mathrm{e}^{\beta\mu_1}+\mathrm{e}^{\beta\mu_2})}\,, \tag{4.19d}$$

$$b_{1,2}^{(2)}(x) = \frac{\phi_-(x-\mathrm{i})\phi_+(x+\mathrm{i})q_1(x+\mathrm{i})q_2(x-\mathrm{i})}{q_1(x-\mathrm{i})q_2(x+\mathrm{i})\widetilde{\Lambda}_1^{(1)}(x)} \cdot \frac{\mathrm{e}^{\beta(\mu_1+\mu_3)}}{\mathrm{e}^{\beta\mu_2}a_1^{(1)}}\,, \tag{4.19e}$$

$$b_{1,3}^{(2)}(x) = \frac{\phi_-(x-\mathrm{i})\phi_+(x+\mathrm{i})q_1(x-2\mathrm{i})}{\phi_-(x-2\mathrm{i})q_2^{(h)}(x)} \cdot \frac{\mathrm{e}^{\beta(\mu_2+\mu_3)}}{\mathrm{e}^{\beta\mu_1}(\mathrm{e}^{\beta\mu_2}+\mathrm{e}^{\beta\mu_3})}\,. \tag{4.19f}$$

The polynomials $q_j^{(h)}(x)$ are defined in analogy to $q_j(x)$, see (2.59), where the Bethe ansatz roots are replaced by the hole solutions of the Bethe ansatz equations. The eigenvalue functions $\widetilde{\Lambda}_1^{(a)}(x)$ are defined via

$$\Lambda_1^{(a)}(x) = n_1^{(a)}(x)\widetilde{\Lambda}_1^{(a)}(x)a_1^{(a)}\,, \tag{4.20}$$

where $n_1^{(a)}(x)$ is the normalization function (3.9), and $a_1^{(a)}$ is the highest coefficient of each polynomial $\Lambda_1^{(a)}(x)$,

$$a_1^{(1)} = \mathrm{e}^{\beta\mu_1}+\mathrm{e}^{\beta\mu_2}+\mathrm{e}^{\beta\mu_3}\,, \qquad a_1^{(2)} = \mathrm{e}^{\beta(\mu_1+\mu_2)}+\mathrm{e}^{\beta(\mu_1+\mu_3)}+\mathrm{e}^{\beta(\mu_2+\mu_3)}\,. \tag{4.21}$$

For the uppercase auxiliary functions, we find

$$B_{1,1}^{(1)}(x) = \frac{q_1(x+\frac{\mathrm{i}}{2})\widetilde{\Lambda}_1^{(1)}(x+\frac{\mathrm{i}}{2})}{\phi_-(x+\frac{\mathrm{i}}{2})q_2^{(h)}(x+\frac{\mathrm{i}}{2})} \cdot \frac{a_1^{(1)}}{\mathrm{e}^{\beta\mu_2}+\mathrm{e}^{\beta\mu_3}}\,, \tag{4.22a}$$

$$B_{1,2}^{(1)}(x) = \frac{q_1^{(h)}(x-\frac{\mathrm{i}}{2})q_2^{(h)}(x+\frac{\mathrm{i}}{2})}{q_1(x+\frac{\mathrm{i}}{2})q_2(x-\frac{\mathrm{i}}{2})\widetilde{\Lambda}_1^{(2)}(x)} \cdot \frac{(\mathrm{e}^{\beta\mu_1}+\mathrm{e}^{\beta\mu_2})(\mathrm{e}^{\beta\mu_2}+\mathrm{e}^{\beta\mu_3})}{a_1^{(2)}}\,, \tag{4.22b}$$

$$B_{1,3}^{(1)}(x) = \frac{q_2(x-\frac{\mathrm{i}}{2})\widetilde{\Lambda}_1^{(1)}(x-\frac{\mathrm{i}}{2})}{\phi_+(x-\frac{\mathrm{i}}{2})q_1^{(h)}(x-\frac{\mathrm{i}}{2})} \cdot \frac{a_1^{(1)}}{\mathrm{e}^{\beta\mu_1}+\mathrm{e}^{\beta\mu_2}}\,, \tag{4.22c}$$

$$B_{1,1}^{(2)}(x) = \frac{q_2(x+\mathrm{i})\widetilde{\Lambda}_1^{(2)}(x+\frac{\mathrm{i}}{2})}{\phi_+(x+2\mathrm{i})q_1^{(h)}(x)} \cdot \frac{a_1^{(2)}}{\mathrm{e}^{\beta\mu_3}(\mathrm{e}^{\beta\mu_1}+\mathrm{e}^{\beta\mu_2})}\,, \tag{4.22d}$$

$$B_{1,2}^{(2)}(x) = \frac{q_1^{(h)}(x)q_2^{(h)}(x)}{q_1(x-\mathrm{i})q_2(x+\mathrm{i})\widetilde{\Lambda}_1^{(1)}(x)} \cdot \frac{(\mathrm{e}^{\beta\mu_1}+\mathrm{e}^{\beta\mu_2})(\mathrm{e}^{\beta\mu_2}+\mathrm{e}^{\beta\mu_3})}{\mathrm{e}^{\beta\mu_2}a_1^{(1)}}\,, \tag{4.22e}$$

$$B_{1,3}^{(2)}(x) = \frac{q_1(x-\mathrm{i})\widetilde{\Lambda}_1^{(2)}(x-\frac{\mathrm{i}}{2})}{\phi_-(x-2\mathrm{i})q_2^{(h)}(x)} \cdot \frac{a_1^{(2)}}{\mathrm{e}^{\beta\mu_1}(\mathrm{e}^{\beta\mu_2}+\mathrm{e}^{\beta\mu_3})}\,. \tag{4.22f}$$

Now, the same procedure using the transform (4.4) as in the last section is applicable, because the number of unknown functions in both cases, $k<0$ and $k>0$, is again equal to the number of uppercase auxiliary functions $\widehat{B}_{1,j}^{(a)}(k)$. However, we will not go into the details of the derivation here, since we will explicitly repeat the arguments for the $sl(4)$-symmetric case in Section 4.2.1, where an even higher number of unknown functions is involved. The result is a self-consistent set of six coupled NLIEs of the form

$$\ln b_{1,j}^{(a)}(x) = -\beta\left(JV^{(a)}(x)+c_j^{(a)}\right) - \sum_{b=1}^{2}\sum_{k=1}^{3}\left[\mathbf{K}_{j,k}^{(a,b)} * \ln B_{1,k}^{(b)}\right](x)\,. \tag{4.23}$$

In the driving terms, we have the functions

$$V^{(1)}(x) = \frac{2\pi}{\sqrt{3}} \frac{1}{2\cosh(2\pi x/3) - 1}\,, \qquad V^{(2)}(x) = \frac{2\pi}{\sqrt{3}} \frac{1}{2\cosh(2\pi x/3) + 1}\,, \tag{4.24}$$

and the constants

$$c_1^{(1)} = (-2\mu_1 + \mu_2 + \mu_3)/3\,, \qquad c_2^{(1)} = (\mu_1 - 2\mu_2 + \mu_3)/3\,, \tag{4.25a}$$
$$c_3^{(1)} = (\mu_1 + \mu_2 - 2\mu_3)/3\,, \qquad c_1^{(2)} = (-\mu_1 - \mu_2 + 2\mu_3)/3\,, \tag{4.25b}$$
$$c_2^{(2)} = (-\mu_1 + 2\mu_2 - \mu_3)/3\,, \qquad c_3^{(2)} = (2\mu_1 - \mu_2 - \mu_3)/3\,. \tag{4.25c}$$

The kernel matrices $\mathbf{K}^{(a,b)}(x)$ are given by

$$\mathbf{K}^{(1,1)}(x) = \begin{pmatrix} K_0(x) & K_1(x) & K_1(x) \\ K_2(x) & K_0(x) & K_1(x) \\ K_2(x) & K_2(x) & K_0(x) \end{pmatrix}\,, \tag{4.26a}$$
$$\mathbf{K}^{(1,2)}(x) = \begin{pmatrix} K_3(x) & K_3(x) & K_4(x) \\ K_3(x) & K_6(x) & K_3(x) \\ K_5(x) & K_3(x) & K_3(x) \end{pmatrix}\,, \tag{4.26b}$$
$$\mathbf{K}^{(2,1)}(x) = \mathbf{K}^{(1,2)}(x)\,, \tag{4.26c}$$
$$\mathbf{K}^{(2,2)}(x) = \mathbf{K}^{(1,1)}(x)\,. \tag{4.26d}$$

The kernel functions are found to be

$$K_0(x) = \mathcal{K}^{(1,1)}(x)\,, \qquad K_1(x) = \mathcal{K}^{(1,1)}(x) + \frac{\mathrm{i}}{x+\mathrm{i}} - \frac{\mathrm{i}}{x-0\mathrm{i}}\,, \tag{4.27a}$$
$$K_2(x) = \mathcal{K}^{(1,1)}(x) + \frac{\mathrm{i}}{x+0\mathrm{i}} - \frac{\mathrm{i}}{x-\mathrm{i}}\,, \qquad K_3(x) = \mathcal{K}^{(1,2)}(x)\,, \tag{4.27b}$$
$$K_4(x) = \mathcal{K}^{(1,2)}(x) + \frac{\mathrm{i}}{x+\frac{3}{2}\mathrm{i}} - \frac{\mathrm{i}}{x+\frac{\mathrm{i}}{2}}\,, \qquad K_5(x) = \mathcal{K}^{(1,2)}(x) + \frac{\mathrm{i}}{x-\frac{\mathrm{i}}{2}} - \frac{\mathrm{i}}{x-\frac{3}{2}\mathrm{i}}\,, \tag{4.27c}$$
$$K_6(x) = \mathcal{K}^{(1,2)}(x) + \frac{\mathrm{i}}{x+\frac{\mathrm{i}}{2}} - \frac{\mathrm{i}}{x-\frac{\mathrm{i}}{2}}\,, \tag{4.27d}$$

with the common transcendental functions

$$\mathcal{K}^{(1,1)}(x) = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln \frac{\Gamma(1+\frac{x}{3}\mathrm{i})\Gamma(\frac{2}{3}-\frac{x}{3}\mathrm{i})}{\Gamma(1-\frac{x}{3}\mathrm{i})\Gamma(\frac{2}{3}+\frac{x}{3}\mathrm{i})}\right)\,, \tag{4.28a}$$
$$\mathcal{K}^{(1,2)}(x) = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln \frac{\Gamma(\frac{1}{6}+\frac{x}{3}\mathrm{i})\Gamma(\frac{1}{2}-\frac{x}{3}\mathrm{i})}{\Gamma(\frac{1}{6}-\frac{x}{3}\mathrm{i})\Gamma(\frac{1}{2}+\frac{x}{3}\mathrm{i})}\right)\,. \tag{4.28b}$$

Finally, the largest eigenvalue of the QTM can be calculated via

$$\ln \Lambda_1^{(1)}(0) = -\beta\left\{J\left(1 - \frac{\pi}{3\sqrt{3}} - \ln 3\right) - \frac{1}{3}\sum_{j=1}^{3}\mu_j\right\} + \sum_{a=1}^{2}\sum_{j=1}^{3}\left[V^{(a)} * \ln B_j^{(a)}\right](0)\,. \tag{4.29}$$

4.1.3 The $sl(2|1)$-symmetric case

Let us now turn to the simplest non-trivial case with different gradings, the $sl(2|1)$-symmetric case of the Uimin-Sutherland model. From the definition of the Hamiltonian (2.2) it is immediately clear that there exist three equivalent formulations, differing only in the choice of grading. In the framework of the Bethe ansatz formulation (2.58), however, this equivalence is not so obvious. We show in Appendix A, how the different formulations can be transformed into one another.

The set of coupled NLIEs for the $sl(2|1)$-symmetric case has first been derived by Jüttner, Klümper and Suzuki [36, 37]. Although the structure of the graded models, for example their representation theory, is generally more difficult, we need only three auxiliary functions here, compared with the six functions of the $sl(3)$-symmetric case. The appearance of the auxiliary functions is simplest for the grading $\epsilon_1 = -\epsilon_2 = \epsilon_3 = +1$, which we will further on denote by $(+-+)$. Here, the functions are actually a subset of those for the $sl(3)$-symmetric case,

$$b^{(1)}_{1,1}(x) = \left.\frac{\boxed{1}}{\boxed{2}+\boxed{3}}\right|_{x+\mathrm{i}/2}\,, \qquad b^{(1)}_{1,2}(x) = \left.\frac{\boxed{3}}{\boxed{1}+\boxed{2}}\right|_{x-\mathrm{i}/2}\,, \tag{4.30a}$$

$$b^{(2)}_{1,1}(x) = \left.\frac{\boxed{1}\cdot\boxed{3}}{\boxed{2}\cdot\left(\boxed{1}+\boxed{2}+\boxed{3}\right)}\right|_{x}\,. \tag{4.30b}$$

After the definition of the uppercase functions $B^{(a)}_{1,j}(x) = b^{(a)}_{1,j}(x) + 1$, we again recognize the connection to the auxiliary functions of the TBA approach,

$$Y^{(1)}_1(x) = B^{(1)}_{1,1}(x) B^{(1)}_{1,2}(x)\,, \qquad Y^{(2)}_1(x) = B^{(2)}_{1,1}(x)\,. \tag{4.31}$$

The latter relation, which of course also means that $y^{(2)}_1(x) = b^{(2)}_{1,1}(x)$, may come as a surprise, as it is not immediately obvious from the definitions. It can, however, easily be checked by cancelling all common Young tableaux in the numerator and denominator of the corresponding TBA auxiliary function. Note also that the former relation bears strong similarity to the $sl(2)$-symmetric case (4.16).

As before, we assure ourselves that the auxiliary functions have the ANZC property in some strip $-1/2 \lesssim \Im(x) \lesssim 1/2$ around the real axis. The factorization of the auxiliary functions yields,

$$b^{(1)}_{1,1}(x) = \frac{\phi_-(x-\frac{\mathrm{i}}{2})\phi_+(x+\frac{\mathrm{i}}{2})q_1(x+\frac{3}{2}\mathrm{i})}{\phi_-(x+\frac{\mathrm{i}}{2})q_2(x-\frac{\mathrm{i}}{2})q^{(h)}_2(x+\frac{\mathrm{i}}{2})} \cdot \frac{\mathrm{e}^{\beta\mu_1}}{\mathrm{e}^{\beta\mu_2}+\mathrm{e}^{\beta\mu_3}}\,, \tag{4.32a}$$

$$b^{(1)}_{1,2}(x) = \frac{\phi_-(x-\frac{\mathrm{i}}{2})\phi_+(x+\frac{\mathrm{i}}{2})q_2(x-\frac{3}{2}\mathrm{i})}{\phi_+(x-\frac{\mathrm{i}}{2})q_1(x+\frac{\mathrm{i}}{2})q^{(h)}_1(x-\frac{\mathrm{i}}{2})} \cdot \frac{\mathrm{e}^{\beta\mu_3}}{\mathrm{e}^{\beta\mu_1}+\mathrm{e}^{\beta\mu_2}}\,, \tag{4.32b}$$

$$b^{(2)}_{1,1}(x) = \frac{\phi_-(x-\mathrm{i})\phi_+(x+\mathrm{i})}{\widetilde{\Lambda}^{(1)}_1(x)} \cdot \frac{\mathrm{e}^{\beta(\mu_1+\mu_3)}}{\mathrm{e}^{\beta\mu_2}a^{(1)}_1}\,, \tag{4.32c}$$

where $a_1^{(1)} = \mathrm{e}^{\beta\mu_1} + \mathrm{e}^{\beta\mu_2} + \mathrm{e}^{\beta\mu_3}$. For the uppercase auxiliary functions, we find

$$B_{1,1}^{(1)}(x) = \frac{q_1(x+\frac{\mathrm{i}}{2})\widetilde{\Lambda}_1^{(1)}(x+\frac{\mathrm{i}}{2})}{\phi_-(x+\frac{\mathrm{i}}{2})q_2(x-\frac{\mathrm{i}}{2})q_2^{(h)}(x+\frac{\mathrm{i}}{2})} \cdot \frac{a_1^{(1)}}{\mathrm{e}^{\beta\mu_2}+\mathrm{e}^{\beta\mu_3}}\,, \tag{4.33a}$$

$$B_{1,2}^{(1)}(x) = \frac{q_2(x-\frac{\mathrm{i}}{2})\widetilde{\Lambda}_1^{(1)}(x-\frac{\mathrm{i}}{2})}{\phi_+(x-\frac{\mathrm{i}}{2})q_1(x+\frac{\mathrm{i}}{2})q_1^{(h)}(x-\frac{\mathrm{i}}{2})} \cdot \frac{a_1^{(1)}}{\mathrm{e}^{\beta\mu_1}+\mathrm{e}^{\beta\mu_2}}\,, \tag{4.33b}$$

$$B_{1,1}^{(2)}(x) = \frac{q_1^{(h)}(x)q_2^{(h)}(x)}{\widetilde{\Lambda}_1^{(1)}(x)} \cdot \frac{(\mathrm{e}^{\beta\mu_1}+\mathrm{e}^{\beta\mu_2})(\mathrm{e}^{\beta\mu_2}+\mathrm{e}^{\beta\mu_3})}{\mathrm{e}^{\beta\mu_2}a_1^{(1)}}\,. \tag{4.33c}$$

Due to the fact that the hole-solution polynomials $q_1^{(h)}(x)$ and $q_2^{(h)}(x)$ are only of degree $N/2$ here, with zeros lying completely above and below the real axis, respectively, we again have the correct number of unknown functions after applying the transform (4.4). The rest of the derivation is straightforward.

Before giving the results, let us also deal with the two other possible choices for the grading. In fact, the auxiliary functions—and therefore the resulting set of coupled NLIEs—are exactly the same in all three possible gradings. However, the role of zeros, poles and the chemical potentials change. From the results of Appendix A, we know how all factors in the factorized form of the auxiliary functions (4.32) and (4.33) have to be renamed in order to change the grading from $(+-+)$ to the case $(-++)$ or $(++-)$. Then, it is easy to regress to the Young tableaux formulation, which indeed looks different in all three cases. For the $(-++)$ case, we find

$$b_{1,1}^{(1)}(x) = \left.\frac{\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\cdot\left(\begin{array}{|c|}\hline 1\\\hline 1\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\right)}{\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\cdot\left(\begin{array}{|c|}\hline 1\\\hline 1\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right)}\right|_{x}\,, \qquad b_{1,2}^{(1)}(x) = \left.\frac{\boxed{3}}{\boxed{1}+\boxed{2}}\right|_{x-\mathrm{i}/2}\,, \tag{4.34a}$$

$$b_{1,1}^{(2)}(x) = \left.\frac{\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 1\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}}\right|_{x-\mathrm{i}/2}\,. \tag{4.34b}$$

Similarly, in the $(++-)$ case, the auxiliary functions have the form

$$b_{1,1}^{(1)}(x) = \left.\frac{\boxed{1}}{\boxed{2}+\boxed{3}}\right|_{x+\mathrm{i}/2}\,, \qquad b_{1,2}^{(1)}(x) = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\cdot\left(\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 3\\\hline\end{array}\right)}{\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 3\\\hline\end{array}\right)}\right|_{x}\,, \tag{4.35a}$$

$$b_{1,1}^{(2)}(x) = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 3\\\hline\end{array}}\right|_{x+\mathrm{i}/2}\,. \tag{4.35b}$$

Note that these formulations also bear a strong resemblance to the $sl(3)$ auxiliary functions, which becomes obvious by deleting all Young tableaux that are not admissible in the $sl(3)$-symmetric case.

The resulting set of three coupled NLIEs is given by

$$\ln b_{1,j}^{(a)}(x) = -\beta\left(JV^{(a)}(x) + c_j^{(a)}\right) - \sum_{b=1}^{2}\sum_{k=1}^{3-b}\left[\mathbf{K}_{j,k}^{(a,b)} * \ln B_k^{(b)}\right](x)\,, \tag{4.36}$$

where the driving terms are

$$V^{(1)}(x) = \frac{4}{4x^2+1}\,, \qquad V^{(2)}(x) = \frac{2}{x^2+1}\,. \tag{4.37}$$

As noted before, only the labelling of the general chemical potentials depends on the choice of grading. Let us stick to the $(+-+)$ case here, for which we find the constants to be

$$c_1^{(1)} = \mu_2 - \mu_1\,, \qquad c_2^{(1)} = \mu_2 - \mu_3\,, \qquad c_1^{(2)} = 2\mu_2 - \mu_1 - \mu_3\,. \tag{4.38}$$

The complete 3-by-3 kernel matrix is

$$\mathbf{K}(x) = \begin{pmatrix} \mathbf{K}^{(1,1)}(x) & \mathbf{K}^{(1,2)}(x) \\ \mathbf{K}^{(2,1)}(x) & \mathbf{K}^{(2,2)}(x) \end{pmatrix} = \left(\begin{array}{cc:c} 0 & K_1(x) & K_3(x) \\ K_2(x) & 0 & K_3(x) \\ \hdashline K_3(x) & K_3(x) & K_4(x) \end{array}\right)\,, \tag{4.39}$$

where the integration kernels have the form

$$K_1(x) = \frac{\mathrm{i}}{x+\mathrm{i}} - \frac{\mathrm{i}}{x-0\mathrm{i}}\,, \qquad K_2(x) = \frac{\mathrm{i}}{x+0\mathrm{i}} - \frac{\mathrm{i}}{x-\mathrm{i}}\,, \tag{4.40a}$$

$$K_3(x) = \frac{4}{4x^2+1}\,, \qquad K_4(x) = \frac{2}{x^2+1}\,, \tag{4.40b}$$

We like to note here, that the matrix $\mathbf{K}^{(1,1)}(x)$ has exactly the same structure as the kernel matrix of the $sl(2)$-symmetric case, despite the fact that there is no common transcendental part here. This may seem coincidental at this stage, but is already a hint at the general structure.

Eventually, the largest eigenvalue of the QTM can be calculated from the uppercase auxiliary functions,

$$\begin{aligned} \ln \Lambda_1^{(1)}(0) &= \beta\,(J+\mu_2) + \sum_{a=1}^{2}\sum_{j=1}^{3-a}\left[V^{(a)} * \ln B_j^{(a)}\right](0) \\ &= -\beta(J+\mu_2-\mu_1-\mu_3) - \ln b_{1,1}^{(2)}(0)\,. \end{aligned} \tag{4.41}$$

Note that the latter form of this expression is a bit special, since the corresponding simplification is only possible for the $sl(2|1)$-symmetric case. In contrast, the former form coincides with the general structure we have already gained from the TBA approach, see equation (3.40).

4.2 New sets of nonlinear integral equations

Now we will show, how the previously gained results can be extended to the four-component cases of the Uimin-Sutherland model. These extensions are the main results of this thesis, together with the numerical evaluation in Chapter 6.

4.2.1 The $sl(4)$-symmetric case

Recall the results for the $sl(2)$- and $sl(3)$-symmetric cases, where the set of auxiliary functions resolves into as many subsets as the number of fundamental algebra representations, and the number of auxiliary functions in each subset is given by the dimension of the fundamental representations. Admitting this structure in general, we expect a total of 14 auxiliary functions for the $sl(4)$-symmetric case, which should be dividable into three subsets. Two of these should consist of four, and one of them of six auxiliary functions. Moreover, the uppercase auxiliary functions should again be connected to those of the TBA approach, in analogy to equations (4.16) and (4.18).

We define the following four auxiliary functions for the first fundamental representation:

$$b^{(1)}_{1,1}(x) = \left.\frac{\boxed{1}}{\boxed{2}+\boxed{3}+\boxed{4}}\right|_{x+\mathrm{i}/2}, \tag{4.42a}$$

$$b^{(1)}_{1,2}(x) = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\cdot\left(\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)}{\left(\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}\right)\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)}\right|_{x}, \tag{4.42b}$$

$$b^{(1)}_{1,3}(x) = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\cdot\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}\cdot\left(\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)}\right|_{x}, \tag{4.42c}$$

$$b^{(1)}_{1,4}(x) = \left.\frac{\boxed{4}}{\boxed{1}+\boxed{2}+\boxed{3}}\right|_{x-\mathrm{i}/2}. \tag{4.42d}$$

We have six auxiliary functions for the second fundamental representation:

$$b^{(2)}_{1,1}(x) = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}}\right|_{x+\mathrm{i}/2}, \tag{4.43a}$$

$$b^{(2)}_{1,2}(x) = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\cdot\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}}{\left(\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)\cdot\left(\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)}\right|_{x+\mathrm{i}/2}, \tag{4.43b}$$

$$b^{(2)}_{1,3}(x) = \left.\frac{\boxed{1}\cdot\boxed{4}}{\left(\boxed{2}+\boxed{3}\right)\cdot\left(\boxed{1}+\boxed{2}+\boxed{3}+\boxed{4}\right)}\right|_{x}, \tag{4.43c}$$

$$b^{(2)}_{1,4}(x) = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\cdot\begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}}{\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}\right)\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}\right)}\right|_{x}, \tag{4.43d}$$

$$b^{(2)}_{1,5}(x) = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\cdot\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}}{\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}\right)\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right)}\right|_{x-\mathrm{i}/2}, \tag{4.43e}$$

$$b^{(2)}_{1,6}(x) = \left.\frac{\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}}\right|_{x-\mathrm{i}/2}. \tag{4.43f}$$

And finally, the four auxiliary functions for the third fundamental representation are:

$$b^{(3)}_{1,1}(x) = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}}\right|_{x+\mathrm{i}/2}, \tag{4.44a}$$

$$b^{(3)}_{1,2}(x) = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\cdot\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}}{\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}\right)}\right|_{x}, \tag{4.44b}$$

$$b^{(3)}_{1,3}(x) = \left.\frac{\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}\right)}{\left(\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}\right)\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)}\right|_{x}, \tag{4.44c}$$

$$b^{(3)}_{1,4}(x) = \left.\frac{\begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}}\right|_{x-\mathrm{i}/2}. \tag{4.44d}$$

The uppercase functions $B^{(a)}_{1,j}(x) = b^{(a)}_{1,j}(x)+1$ are defined as usual. They can also be written in a form, where only simple sums of Young tableaux appear as factors in the numerators.

For the first subset, we get

$$B^{(1)}_{1,1}(x) = \left.\frac{\boxed{1}+\boxed{2}+\boxed{3}+\boxed{4}}{\boxed{2}+\boxed{3}+\boxed{4}}\right|_{x+\mathrm{i}/2}, \tag{4.45a}$$

$$B^{(1)}_{1,2}(x) = \left.\frac{\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}\right)\cdot\left(\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)}{\left(\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}\right)\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)}\right|_{x}, \tag{4.45b}$$

$$B^{(1)}_{1,3}(x) = \left.\frac{\left(\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}\right)\cdot\left(\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)}{\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}\cdot\left(\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)}\right|_{x}, \tag{4.45c}$$

$$B^{(1)}_{1,4}(x) = \left.\frac{\boxed{1}+\boxed{2}+\boxed{3}+\boxed{4}}{\boxed{1}+\boxed{2}+\boxed{3}}\right|_{x-\mathrm{i}/2}. \tag{4.45d}$$

The uppercase functions of the second subset are

$$B^{(2)}_{1,1}(x) = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}}\right|_{x+\mathrm{i}/2}, \tag{4.46a}$$

$$B^{(2)}_{1,2}(x) = \left.\frac{\left(\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)\cdot\left(\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)}{\left(\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)\cdot\left(\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)}\right|_{x+\mathrm{i}/2}, \tag{4.46b}$$

$$B^{(2)}_{1,3}(x) = \left.\frac{\left(\boxed{1}+\boxed{2}+\boxed{3}\right)\cdot\left(\boxed{2}+\boxed{3}+\boxed{4}\right)}{\left(\boxed{2}+\boxed{3}\right)\cdot\left(\boxed{1}+\boxed{2}+\boxed{3}+\boxed{4}\right)}\right|_{x}, \tag{4.46c}$$

$$B^{(2)}_{1,4}(x) = \left.\frac{\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}\right)\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}\right)}{\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}\right)\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}\right)}\right|_{x}, \tag{4.46d}$$

$$B^{(2)}_{1,5}(x) = \left.\frac{\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\right)\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}\right)}{\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}\right)\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right)}\right|_{x-\mathrm{i}/2}, \tag{4.46e}$$

$$B^{(2)}_{1,6}(x) = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}}\right|_{x-\mathrm{i}/2}. \tag{4.46f}$$

For the third subset, the results are

$$B^{(3)}_{1,1}(x) = \left. \frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}} \right|_{x+\mathrm{i}/2} , \tag{4.47a}$$

$$B^{(3)}_{1,2}(x) = \left. \frac{\left(\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}\right) \cdot \left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right)}{\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} \cdot \left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}\right)} \right|_{x} , \tag{4.47b}$$

$$B^{(3)}_{1,3}(x) = \left. \frac{\left(\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right) \cdot \left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}\right)}{\left(\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}\right) \cdot \left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)} \right|_{x} , \tag{4.47c}$$

$$B^{(3)}_{1,4}(x) = \left. \frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}} \right|_{x-\mathrm{i}/2} . \tag{4.47d}$$

Using the explicit form of the functions $B^{(a)}_{1,j}(x)$, it can easily be shown that the relations

$$Y^{(a)}_1(x) = \prod_{j=1}^{d_a} B^{(a)}_{1,j}(x) \tag{4.48}$$

hold as expected, where $d_a = \binom{4}{a}$ is the dimension of the ath fundamental representation of $sl(4)$. Like in the $sl(3)$-symmetric case, the set of auxiliary functions (4.42) is related to (4.44) by a conjugation transformation to the Young tableaux. The set (4.43) is self-conjugate in this sense. Note that a second set of valid auxiliary functions for the $sl(4)$-symmetric case can actually be obtained by a complex conjugation of all auxiliary functions.

Let us take a closer look at the sums of Young tableaux appearing as factors in the numerators and denominators of the auxiliary functions. It is a remarkable fact that all these factors are partial sums of the Young tableaux appearing in the three eigenvalue functions,

$$\Lambda^{(1)}_1(x) = \left. \boxed{1} + \boxed{2} + \boxed{3} + \boxed{4} \right|_x , \tag{4.49a}$$

$$\Lambda^{(2)}_1(x) = \left. \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array} \right|_x , \tag{4.49b}$$

$$\Lambda_1^{(3)}(x) = \left.\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}\right\|_x . \tag{4.49c}$$

Therefore, several of the potential poles in the summands of each term vanish. We find the explicit terms

$$\left.\boxed{2} + \boxed{3}\right\|_x = \frac{\phi_-(x)\phi_+(x)q_2^{(h)}(x)}{q_1(x)q_3(x)} \cdot \vartheta_2 \,, \tag{4.50a}$$

$$\left.\boxed{3} + \boxed{4}\right\|_x = \frac{\phi_-(x)q_3^{(h)}(x)}{q_2(x)} \cdot \vartheta_3 \,, \tag{4.50b}$$

$$\left.\boxed{1} + \boxed{2} + \boxed{3}\right\|_x = \frac{\phi_+(x)X_1^{(1)}(x)}{q_3(x)} \cdot \chi_1^{(1)} \,, \tag{4.50c}$$

$$\left.\boxed{2} + \boxed{3} + \boxed{4}\right\|_x = \frac{\phi_-(x)X_2^{(1)}(x)}{q_1(x)} \cdot \chi_2^{(1)} \,, \tag{4.50d}$$

$$\left.\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}\right\|_x = \frac{\phi_-(x+\frac{\mathrm{i}}{2})\phi_+(x-\frac{\mathrm{i}}{2})X_1^{(2)}(x)}{q_2(x-\frac{\mathrm{i}}{2})} \cdot \chi_1^{(2)} \,, \tag{4.50e}$$

$$\left.\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right\|_x = \frac{\phi_-(x+\frac{\mathrm{i}}{2})\phi_+(x-\frac{\mathrm{i}}{2})X_2^{(2)}(x)}{q_2(x+\frac{\mathrm{i}}{2})} \cdot \chi_2^{(2)} \,, \tag{4.50f}$$

$$\left.\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right\|_x = \frac{\phi_-(x+\frac{\mathrm{i}}{2})\phi_+(x-\frac{\mathrm{i}}{2})\phi_+(x+\frac{\mathrm{i}}{2})X_1^{(3)}(x)}{q_3(x+\frac{\mathrm{i}}{2})} \cdot \chi_1^{(3)} \,, \tag{4.50g}$$

$$\left.\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right\|_x = \frac{\phi_-(x+\frac{\mathrm{i}}{2})\phi_+(x-\frac{\mathrm{i}}{2})\phi_-(x-\frac{\mathrm{i}}{2})X_2^{(3)}(x)}{q_1(x-\frac{\mathrm{i}}{2})} \cdot \chi_2^{(3)} \,, \tag{4.50h}$$

where the functions $q_j^{(h)}(x)$ and $X_j^{(a)}(x)$ are polynomials of degree N, with the exception of $X_1^{(2)}(x)$ and $X_2^{(2)}(x)$ being of degree $3N/2$. We also use the normalized eigenvalue functions $\widetilde{\Lambda}_1^{(b)}(x)$ again, where $\Lambda_1^{(b)}(x) = n_1^{(b)}(x)\widetilde{\Lambda}_1^{(b)}(x)a_1^{(b)}$. The constants

$$\vartheta_2 = \mathrm{e}^{\beta\mu_2} + \mathrm{e}^{\beta\mu_3} \,, \tag{4.51a}$$

$$\vartheta_3 = \mathrm{e}^{\beta\mu_3} + \mathrm{e}^{\beta\mu_4} \,, \tag{4.51b}$$

$$\chi_1^{(1)} = \mathrm{e}^{\beta\mu_1} + \mathrm{e}^{\beta\mu_2} + \mathrm{e}^{\beta\mu_3} \,, \tag{4.51c}$$

$$\chi_2^{(1)} = \mathrm{e}^{\beta\mu_2} + \mathrm{e}^{\beta\mu_3} + \mathrm{e}^{\beta\mu_4} \,, \tag{4.51d}$$

$$\chi_1^{(2)} = \mathrm{e}^{\beta(\mu_1+\mu_2)} + \mathrm{e}^{\beta(\mu_1+\mu_3)} + \mathrm{e}^{\beta(\mu_1+\mu_4)} + \mathrm{e}^{\beta(\mu_2+\mu_3)} + \mathrm{e}^{\beta(\mu_2+\mu_4)} \,, \tag{4.51e}$$

$$\chi_2^{(2)} = \mathrm{e}^{\beta(\mu_1+\mu_3)} + \mathrm{e}^{\beta(\mu_1+\mu_4)} + \mathrm{e}^{\beta(\mu_2+\mu_3)} + \mathrm{e}^{\beta(\mu_2+\mu_4)} + \mathrm{e}^{\beta(\mu_3+\mu_4)} \,, \tag{4.51f}$$

$$\chi_1^{(3)} = \mathrm{e}^{\beta(\mu_1+\mu_2)} + \mathrm{e}^{\beta(\mu_1+\mu_3)} + \mathrm{e}^{\beta(\mu_2+\mu_3)} \,, \tag{4.51g}$$

$$\chi_2^{(3)} = \mathrm{e}^{\beta(\mu_2+\mu_3)} + \mathrm{e}^{\beta(\mu_2+\mu_4)} + \mathrm{e}^{\beta(\mu_3+\mu_4)} \,, \tag{4.51h}$$

$$a_1^{(1)} = \mathrm{e}^{\beta\mu_1} + \mathrm{e}^{\beta\mu_2} + \mathrm{e}^{\beta\mu_3} + \mathrm{e}^{\beta\mu_4}\,, \tag{4.51i}$$
$$a_1^{(2)} = \mathrm{e}^{\beta(\mu_1+\mu_2)} + \mathrm{e}^{\beta(\mu_1+\mu_3)} + \mathrm{e}^{\beta(\mu_1+\mu_4)} + \mathrm{e}^{\beta(\mu_2+\mu_3)} + \mathrm{e}^{\beta(\mu_2+\mu_4)} + \mathrm{e}^{\beta(\mu_3+\mu_4)}\,, \tag{4.51j}$$
$$a_1^{(3)} = \mathrm{e}^{\beta(\mu_1+\mu_2+\mu_3)} + \mathrm{e}^{\beta(\mu_1+\mu_2+\mu_4)} + \mathrm{e}^{\beta(\mu_1+\mu_3+\mu_4)} + \mathrm{e}^{\beta(\mu_2+\mu_3+\mu_4)} \tag{4.51k}$$

have been separated, so that the highest coefficient of each polynomial is one. Again, the roots of the polynomials $q_j^{(h)}(x)$ are given by the hole solutions of the Bethe ansatz equations (2.60). The roots of the other polynomials, however, are new and have no particular meaning in terms of the Bethe ansatz. From numerical solutions of the Bethe ansatz equations at finite Trotter number N, we know the root structure of all emergent functions. They are located in groups of $N/2$ many roots on slightly curved lines with imaginary parts close to the values

$$q_j(x)\colon 0\,, \qquad \widetilde{\Lambda}_1^{(a)}(x)\colon \pm(a+1)/2\,, \tag{4.52a}$$
$$q_j^{(h)}(x)\colon \pm 1\,, \qquad X_j^{(1)}(x)\colon \pm 1\,, \tag{4.52b}$$
$$X_1^{(2)}(x)\colon +1/2,\ \pm 3/2\,, \qquad X_2^{(2)}(x)\colon -1/2,\ \pm 3/2\,, \tag{4.52c}$$
$$X_j^{(3)}(x)\colon \pm 3/2\,. \tag{4.52d}$$

Using the information on the constituents, we are able to write down the auxiliary functions in factorized form,

$$b_{1,1}^{(1)}(x) = \frac{\phi_-(x-\frac{\mathrm{i}}{2})\phi_+(x+\frac{\mathrm{i}}{2})q_1(x+\frac{3}{2}\mathrm{i})}{\phi_-(x+\frac{\mathrm{i}}{2})X_2^{(1)}(x+\frac{\mathrm{i}}{2})} \cdot \frac{\mathrm{e}^{\beta\mu_1}}{\chi_2^{(1)}}\,, \tag{4.53a}$$
$$b_{1,2}^{(1)}(x) = \frac{\phi_-(x-\frac{\mathrm{i}}{2})\phi_+(x+\frac{\mathrm{i}}{2})q_2(x+\frac{3}{2}\mathrm{i})X_2^{(3)}(x)}{q_1(x+\frac{\mathrm{i}}{2})q_3^{(h)}(x+\frac{\mathrm{i}}{2})\widetilde{\Lambda}_1^{(2)}(x)} \cdot \frac{\mathrm{e}^{\beta\mu_2}\chi_2^{(3)}}{\vartheta_3 a_1^{(2)}}\,, \tag{4.53b}$$
$$b_{1,3}^{(1)}(x) = \frac{\phi_-(x-\frac{\mathrm{i}}{2})\phi_+(x+\frac{\mathrm{i}}{2})q_2(x-\frac{3}{2}\mathrm{i})q_3(x+\frac{3}{2}\mathrm{i})}{q_3(x-\frac{\mathrm{i}}{2})X_2^{(2)}(x)} \cdot \frac{\mathrm{e}^{2\beta\mu_3}}{\chi_2^{(2)}}\,, \tag{4.53c}$$
$$b_{1,4}^{(1)}(x) = \frac{\phi_-(x-\frac{\mathrm{i}}{2})\phi_+(x+\frac{\mathrm{i}}{2})q_3(x-\frac{3}{2}\mathrm{i})}{\phi_+(x-\frac{\mathrm{i}}{2})X_1^{(1)}(x-\frac{\mathrm{i}}{2})} \cdot \frac{\mathrm{e}^{\beta\mu_4}}{\chi_1^{(1)}}\,, \tag{4.53d}$$
$$b_{1,1}^{(2)}(x) = \frac{\phi_-(x-\mathrm{i})\phi_+(x+\mathrm{i})q_2(x+2\mathrm{i})}{X_2^{(2)}(x+\frac{\mathrm{i}}{2})} \cdot \frac{\mathrm{e}^{\beta(\mu_1+\mu_2)}}{\chi_2^{(2)}}\,, \tag{4.53e}$$
$$b_{1,2}^{(2)}(x) = \frac{\phi_-(x-\mathrm{i})\phi_+(x+\mathrm{i})q_1(x+\mathrm{i})q_2(x-\mathrm{i})q_3(x+2\mathrm{i})}{q_2(x+\mathrm{i})X_1^{(1)}(x)X_2^{(3)}(x+\frac{\mathrm{i}}{2})} \cdot \frac{\mathrm{e}^{\beta(\mu_1+2\mu_3)}}{\chi_1^{(1)}\chi_2^{(3)}}\,, \tag{4.53f}$$
$$b_{1,3}^{(2)}(x) = \frac{\phi_-(x-\mathrm{i})\phi_+(x+\mathrm{i})q_1(x+\mathrm{i})q_3(x-\mathrm{i})}{q_2^{(h)}(x)\widetilde{\Lambda}_1^{(1)}(x)} \cdot \frac{\mathrm{e}^{\beta(\mu_1+\mu_4)}}{\vartheta_2 a_1^{(1)}}\,, \tag{4.53g}$$
$$b_{1,4}^{(2)}(x) = \frac{\phi_-(x-\mathrm{i})\phi_+(x+\mathrm{i})q_1(x-2\mathrm{i})q_3(x+2\mathrm{i})}{q_2^{(h)}(x)\widetilde{\Lambda}_1^{(3)}(x)} \cdot \frac{\mathrm{e}^{2\beta(\mu_2+\mu_3)}}{\vartheta_2 a_1^{(3)}}\,, \tag{4.53h}$$

$$b_{1,5}^{(2)}(x) = \frac{\phi_-(x-\mathrm{i})\phi_+(x+\mathrm{i})q_1(x-2\mathrm{i})q_2(x+\mathrm{i})q_3(x-\mathrm{i})}{q_2(x-\mathrm{i})X_2^{(1)}(x)X_1^{(3)}(x-\frac{\mathrm{i}}{2})} \cdot \frac{\mathrm{e}^{\beta(2\mu_2+\mu_4)}}{\chi_2^{(1)}\chi_1^{(3)}}\,, \tag{4.53i}$$

$$b_{1,6}^{(2)}(x) = \frac{\phi_-(x-\mathrm{i})\phi_+(x+\mathrm{i})q_2(x-2\mathrm{i})}{X_1^{(2)}(x-\frac{\mathrm{i}}{2})} \cdot \frac{\mathrm{e}^{\beta(\mu_3+\mu_4)}}{\chi_1^{(2)}}\,, \tag{4.53j}$$

$$b_{1,1}^{(3)}(x) = \frac{\phi_-(x-\frac{3}{2}\mathrm{i})\phi_+(x+\frac{3}{2}\mathrm{i})q_3(x+\frac{5}{2}\mathrm{i})}{\phi_+(x+\frac{5}{2}\mathrm{i})X_1^{(3)}(x)} \cdot \frac{\mathrm{e}^{\beta(\mu_1+\mu_2+\mu_3)}}{\mathrm{e}^{\beta\mu_4}\chi_1^{(3)}}\,, \tag{4.53k}$$

$$b_{1,2}^{(3)}(x) = \frac{\phi_-(x-\frac{3}{2}\mathrm{i})\phi_+(x+\frac{3}{2}\mathrm{i})q_2(x+\frac{3}{2}\mathrm{i})q_3(x-\frac{\mathrm{i}}{2})}{q_3(x+\frac{3}{2}\mathrm{i})X_1^{(2)}(x)} \cdot \frac{\mathrm{e}^{\beta(\mu_1+\mu_2+\mu_4)}}{\mathrm{e}^{\beta\mu_3}\chi_1^{(2)}}\,, \tag{4.53l}$$

$$b_{1,3}^{(3)}(x) = \frac{\phi_-(x-\frac{3}{2}\mathrm{i})\phi_+(x+\frac{3}{2}\mathrm{i})q_2(x-\frac{3}{2}\mathrm{i})X_2^{(1)}(x+\frac{\mathrm{i}}{2})}{q_1(x-\frac{3}{2}\mathrm{i})q_3^{(h)}(x+\frac{\mathrm{i}}{2})\widetilde{\Lambda}_1^{(2)}(x)} \cdot \frac{\mathrm{e}^{\beta(\mu_1+\mu_3+\mu_4)}\chi_2^{(1)}}{\mathrm{e}^{\beta\mu_2}\vartheta_3 a_1^{(2)}}\,, \tag{4.53m}$$

$$b_{1,4}^{(3)}(x) = \frac{\phi_-(x-\frac{3}{2}\mathrm{i})\phi_+(x+\frac{3}{2}\mathrm{i})q_1(x-\frac{5}{2}\mathrm{i})}{\phi_-(x-\frac{5}{2}\mathrm{i})X_2^{(3)}(x)} \cdot \frac{\mathrm{e}^{\beta(\mu_2+\mu_3+\mu_4)}}{\mathrm{e}^{\beta\mu_1}\chi_2^{(3)}}\,. \tag{4.53n}$$

For the uppercase auxiliary functions, we get

$$B_{1,1}^{(1)}(x) = \frac{q_1(x+\frac{\mathrm{i}}{2})\widetilde{\Lambda}_1^{(1)}(x+\frac{\mathrm{i}}{2})}{\phi_-(x+\frac{\mathrm{i}}{2})X_2^{(1)}(x+\frac{\mathrm{i}}{2})} \cdot \frac{a_1^{(1)}}{\chi_2^{(1)}}\,, \tag{4.54a}$$

$$B_{1,2}^{(1)}(x) = \frac{X_2^{(1)}(x+\frac{\mathrm{i}}{2})X_2^{(2)}(x)}{q_1(x+\frac{\mathrm{i}}{2})q_3^{(h)}(x+\frac{\mathrm{i}}{2})\widetilde{\Lambda}_1^{(2)}(x)} \cdot \frac{\chi_2^{(1)}\chi_2^{(2)}}{\vartheta_3 a_1^{(2)}}\,, \tag{4.54b}$$

$$B_{1,3}^{(1)}(x) = \frac{q_3^{(h)}(x+\frac{\mathrm{i}}{2})X_1^{(1)}(x-\frac{\mathrm{i}}{2})}{q_3(x-\frac{\mathrm{i}}{2})X_2^{(2)}(x)} \cdot \frac{\vartheta_3\chi_1^{(1)}}{\chi_2^{(2)}}\,, \tag{4.54c}$$

$$B_{1,4}^{(1)}(x) = \frac{q_3(x-\frac{\mathrm{i}}{2})\widetilde{\Lambda}_1^{(1)}(x-\frac{\mathrm{i}}{2})}{\phi_+(x-\frac{\mathrm{i}}{2})X_1^{(1)}(x-\frac{\mathrm{i}}{2})} \cdot \frac{a_1^{(1)}}{\chi_1^{(1)}}\,, \tag{4.54d}$$

$$B_{1,1}^{(2)}(x) = \frac{q_2(x+\mathrm{i})\widetilde{\Lambda}_1^{(2)}(x+\frac{\mathrm{i}}{2})}{X_2^{(2)}(x+\frac{\mathrm{i}}{2})} \cdot \frac{a_1^{(2)}}{\chi_2^{(2)}}\,, \tag{4.54e}$$

$$B_{1,2}^{(2)}(x) = \frac{q_2^{(h)}(x)X_2^{(2)}(x+\frac{\mathrm{i}}{2})}{q_2(x+\mathrm{i})X_1^{(1)}(x)X_2^{(3)}(x+\frac{\mathrm{i}}{2})} \cdot \frac{\vartheta_2\chi_2^{(2)}}{\chi_1^{(1)}\chi_2^{(3)}}\,, \tag{4.54f}$$

$$B_{1,3}^{(2)}(x) = \frac{X_1^{(1)}(x)X_2^{(1)}(x)}{q_2^{(h)}(x)\widetilde{\Lambda}_1^{(1)}(x)} \cdot \frac{\chi_1^{(1)}\chi_2^{(1)}}{\vartheta_2 a_1^{(1)}}\,, \tag{4.54g}$$

$$B_{1,4}^{(2)}(x) = \frac{X_1^{(3)}(x-\frac{\mathrm{i}}{2})X_2^{(3)}(x+\frac{\mathrm{i}}{2})}{q_2^{(h)}(x)\widetilde{\Lambda}_1^{(3)}(x)} \cdot \frac{\chi_1^{(3)}\chi_2^{(3)}}{\vartheta_2 a_1^{(3)}}\,, \tag{4.54h}$$

$$B_{1,5}^{(2)}(x) = \frac{q_2^{(h)}(x)X_1^{(2)}(x-\frac{\mathrm{i}}{2})}{q_2(x-\mathrm{i})X_2^{(1)}(x)X_1^{(3)}(x-\frac{\mathrm{i}}{2})} \cdot \frac{\vartheta_2\chi_1^{(2)}}{\chi_2^{(1)}\chi_1^{(3)}}\,, \tag{4.54i}$$

$$B_{1,6}^{(2)}(x) = \frac{q_2(x-\mathrm{i})\widetilde{\Lambda}_1^{(2)}(x-\frac{\mathrm{i}}{2})}{X_1^{(2)}(x-\frac{\mathrm{i}}{2})} \cdot \frac{a_1^{(2)}}{\chi_1^{(2)}}\,, \tag{4.54j}$$

$$B_{1,1}^{(3)}(x) = \frac{q_3(x+\frac{3}{2}\mathrm{i})\widetilde{\Lambda}_1^{(3)}(x+\frac{\mathrm{i}}{2})}{\phi_+(x+\frac{5}{2}\mathrm{i})X_1^{(3)}(x)} \cdot \frac{a_1^{(3)}}{\mathrm{e}^{\beta\mu_4}\chi_1^{(3)}}\,, \tag{4.54k}$$

$$B_{1,2}^{(3)}(x) = \frac{q_3^{(h)}(x+\frac{\mathrm{i}}{2})X_1^{(3)}(x)}{q_3(x+\frac{3}{2}\mathrm{i})X_1^{(2)}(x)} \cdot \frac{\vartheta_3\chi_1^{(3)}}{\mathrm{e}^{\beta\mu_3}\chi_1^{(2)}}\,, \tag{4.54l}$$

$$B_{1,3}^{(3)}(x) = \frac{X_1^{(2)}(x)X_2^{(3)}(x)}{q_1(x-\frac{3}{2}\mathrm{i})q_3^{(h)}(x+\frac{\mathrm{i}}{2})\widetilde{\Lambda}_1^{(2)}(x)} \cdot \frac{\chi_1^{(2)}\chi_2^{(3)}}{\mathrm{e}^{\beta\mu_2}\vartheta_3 a_1^{(2)}}\,, \tag{4.54m}$$

$$B_{1,4}^{(3)}(x) = \frac{q_1(x-\frac{3}{2}\mathrm{i})\widetilde{\Lambda}_1^{(3)}(x-\frac{\mathrm{i}}{2})}{\phi_-(x-\frac{5}{2}\mathrm{i})X_2^{(3)}(x)} \cdot \frac{a_1^{(3)}}{\mathrm{e}^{\beta\mu_1}\chi_2^{(3)}}\,. \tag{4.54n}$$

From the explicit form of the auxiliary functions we can now easily read off the analyticity properties. Again, we find each function to have the ANZC property in the vicinity of the real axis, $-1/2 \lesssim \Im(x) \lesssim 1/2$.

Having analyzed the properties of the auxiliary functions, we are ready to derive the set of NLIEs, again using the standard procedure. Since, however, the corresponding calculations are rather lengthy, details are deferred to Appendix C.

We finally obtain the result

$$\ln b_{1,j}^{(a)}(x) = -\beta\left(JV_{[4]}^{(a)}(x) + c_j^{(a)}\right) - \sum_{b=1}^{3}\sum_{k=1}^{d_b}\left[\mathbf{K}_{j,k}^{(a,b)} * \ln B_{1,k}^{(b)}\right](x)\,, \tag{4.55}$$

which is exactly of the same structure we already found in all previous cases. The functions

$$V_{[n]}^{(a)}(x) = \frac{2\pi}{n}\frac{\sin(\pi a/n)}{\cosh(2\pi x/n) - \cos(\pi a/n)} \tag{4.56}$$

are exactly the inverse Fourier transforms of (C.4), and the kernel matrices are the inverse Fourier transforms of (C.5) and (C.6). Therefore, we get

$$\mathbf{K}^{(1,1)}(x) = \begin{pmatrix} K_0(x) & K_1(x) & K_1(x) & K_1(x) \\ K_2(x) & K_0(x) & K_1(x) & K_1(x) \\ K_2(x) & K_2(x) & K_0(x) & K_1(x) \\ K_2(x) & K_2(x) & K_2(x) & K_0(x) \end{pmatrix}, \tag{4.57a}$$

$$\mathbf{K}^{(2,2)}(x) = \begin{pmatrix} K_3(x) & K_4(x) & K_4(x) & K_4(x) & K_4(x) & K_6(x) \\ K_5(x) & K_3(x) & K_4(x) & K_4(x) & K_8(x) & K_4(x) \\ K_5(x) & K_5(x) & K_3(x) & K_{10}(x) & K_4(x) & K_4(x) \\ K_5(x) & K_5(x) & K_{10}(x) & K_3(x) & K_4(x) & K_4(x) \\ K_5(x) & K_9(x) & K_5(x) & K_5(x) & K_3(x) & K_4(x) \\ K_7(x) & K_5(x) & K_5(x) & K_5(x) & K_5(x) & K_3(x) \end{pmatrix}, \tag{4.57b}$$

$$\mathbf{K}^{(1,2)}(x) = \begin{pmatrix} K_{11}(x) & K_{11}(x) & K_{11}(x) & K_{12}(x) & K_{12}(x) & K_{12}(x) \\ K_{11}(x) & K_{14}(x) & K_{14}(x) & K_{11}(x) & K_{11}(x) & K_{12}(x) \\ K_{13}(x) & K_{11}(x) & K_{14}(x) & K_{11}(x) & K_{14}(x) & K_{11}(x) \\ K_{13}(x) & K_{13}(x) & K_{11}(x) & K_{13}(x) & K_{11}(x) & K_{11}(x) \end{pmatrix}, \tag{4.57c}$$

$$\mathbf{K}^{(1,3)}(x) = \begin{pmatrix} K_{15}(x) & K_{15}(x) & K_{15}(x) & K_{16}(x) \\ K_{15}(x) & K_{15}(x) & K_{18}(x) & K_{15}(x) \\ K_{15}(x) & K_{19}(x) & K_{15}(x) & K_{15}(x) \\ K_{17}(x) & K_{15}(x) & K_{15}(x) & K_{15}(x) \end{pmatrix}, \tag{4.57d}$$

and

$$\mathbf{K}^{(3,3)}(x) = \mathbf{K}^{(1,1)}(x)\,, \qquad \mathbf{K}^{(2,1)}(x) = \left(\mathbf{K}^{(1,2)}(x)\right)^{\dagger}\,, \tag{4.58a}$$

$$\mathbf{K}^{(3,1)}(x) = \left(\mathbf{K}^{(1,3)}(x)\right)^{\dagger}\,, \qquad \mathbf{K}^{(3,2)}(x) = \left(\mathbf{K}^{(2,3)}(x)\right)^{\dagger}\,, \tag{4.58b}$$

$$\mathbf{K}^{(2,3)}_{j,k}(x) = \mathbf{K}^{(1,2)}_{5-k,7-j}(x)\,. \tag{4.58c}$$

The matrices may also be viewed as submatrices of one big kernel matrix

$$\mathbf{K}(x) = \begin{pmatrix} \mathbf{K}^{(1,1)}(x) & \mathbf{K}^{(1,2)}(x) & \mathbf{K}^{(1,3)}(x) \\ \mathbf{K}^{(2,1)}(x) & \mathbf{K}^{(2,2)}(x) & \mathbf{K}^{(2,3)}(x) \\ \mathbf{K}^{(3,1)}(x) & \mathbf{K}^{(3,2)}(x) & \mathbf{K}^{(3,3)}(x) \end{pmatrix}, \tag{4.59}$$

which then features two interesting symmetry relations. Namely, the matrix $\mathbf{K}(x)$ is Hermitian and invariant under a reflection along the antidiagonal,

$$\mathbf{K}(x) = (\mathbf{K}(x))^{\dagger}\,, \qquad K_{j,k}(x) = K_{15-k,15-j}(x)\,. \tag{4.60}$$

The kernel functions are

$$K_0(x) = \mathcal{K}^{(1,1)}_{[4]}(x)\,, \qquad K_1(x) = \mathcal{K}^{(1,1)}_{[4]}(x) + \frac{\mathrm{i}}{x+\mathrm{i}} - \frac{\mathrm{i}}{x-0\mathrm{i}}\,, \tag{4.61a}$$

$$K_2(x) = \mathcal{K}^{(1,1)}_{[4]}(x) + \frac{\mathrm{i}}{x+0\mathrm{i}} - \frac{\mathrm{i}}{x-\mathrm{i}}\,, \qquad K_3(x) = \mathcal{K}^{(2,2)}_{[4]}(x)\,, \tag{4.61b}$$

$$K_4(x) = \mathcal{K}^{(2,2)}_{[4]}(x) + \frac{\mathrm{i}}{x+\mathrm{i}} - \frac{\mathrm{i}}{x-0\mathrm{i}}\,, \qquad K_5(x) = \mathcal{K}^{(2,2)}_{[4]}(x) + \frac{\mathrm{i}}{x+0\mathrm{i}} - \frac{\mathrm{i}}{x-\mathrm{i}}\,, \tag{4.61c}$$

$$K_6(x) = \mathcal{K}^{(2,2)}_{[4]}(x) + \frac{\mathrm{i}}{x+2\mathrm{i}} - \frac{\mathrm{i}}{x-0\mathrm{i}}\,, \qquad K_7(x) = \mathcal{K}^{(2,2)}_{[4]}(x) + \frac{\mathrm{i}}{x+0\mathrm{i}} - \frac{\mathrm{i}}{x-2\mathrm{i}}\,, \tag{4.61d}$$

$$K_8(x) = \mathcal{K}^{(2,2)}_{[4]}(x) + \frac{2\mathrm{i}}{x+\mathrm{i}} - \frac{2\mathrm{i}}{x-0\mathrm{i}}\,, \qquad K_9(x) = \mathcal{K}^{(2,2)}_{[4]}(x) + \frac{2\mathrm{i}}{x+0\mathrm{i}} - \frac{2\mathrm{i}}{x-\mathrm{i}}\,, \tag{4.61e}$$

$$K_{10}(x) = \mathcal{K}^{(2,2)}_{[4]}(x) + \frac{\mathrm{i}}{x+\mathrm{i}} - \frac{\mathrm{i}}{x-\mathrm{i}}\,, \qquad K_{11}(x) = \mathcal{K}^{(1,2)}_{[4]}(x)\,, \tag{4.61f}$$

$$K_{12}(x) = \mathcal{K}^{(1,2)}_{[4]}(x) + \frac{\mathrm{i}}{x+\frac{3}{2}\mathrm{i}} - \frac{\mathrm{i}}{x+\frac{\mathrm{i}}{2}}\,, \qquad K_{13}(x) = \mathcal{K}^{(1,2)}_{[4]}(x) + \frac{\mathrm{i}}{x-\frac{\mathrm{i}}{2}} - \frac{\mathrm{i}}{x-\frac{3}{2}\mathrm{i}}\,, \tag{4.61g}$$

$$K_{14}(x) = \mathcal{K}^{(1,2)}_{[4]}(x) + \frac{\mathrm{i}}{x+\frac{\mathrm{i}}{2}} - \frac{\mathrm{i}}{x-\frac{\mathrm{i}}{2}}\,, \qquad K_{15}(x) = \mathcal{K}^{(1,3)}_{[4]}(x)\,, \tag{4.61h}$$

$$K_{16}(x) = \mathcal{K}^{(1,3)}_{[4]}(x) + \frac{\mathrm{i}}{x+2\mathrm{i}} - \frac{\mathrm{i}}{x+\mathrm{i}}\,, \qquad K_{17}(x) = \mathcal{K}^{(1,3)}_{[4]}(x) + \frac{\mathrm{i}}{x-\mathrm{i}} - \frac{\mathrm{i}}{x-2\mathrm{i}}\,, \tag{4.61i}$$

$$K_{18}(x) = \mathcal{K}^{(1,3)}_{[4]}(x) + \frac{\mathrm{i}}{x+\mathrm{i}} - \frac{\mathrm{i}}{x+0\mathrm{i}}\,, \qquad K_{19}(x) = \mathcal{K}^{(1,3)}_{[4]}(x) + \frac{\mathrm{i}}{x+0\mathrm{i}} - \frac{\mathrm{i}}{x-\mathrm{i}}\,, \tag{4.61j}$$

where the common terms are given by the transcendental functions

$$\mathcal{K}^{(a,a)}_{[n]}(x) = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{j=0}^{a-1} \ln \frac{\Gamma\left(\delta_{j,0} + \frac{j}{n} + \frac{x}{n}\mathrm{i}\right)\Gamma\left(1 + \frac{j-a}{n} - \frac{x}{n}\mathrm{i}\right)}{\Gamma\left(\delta_{j,0} + \frac{j}{n} - \frac{x}{n}\mathrm{i}\right)\Gamma\left(1 + \frac{j-a}{n} + \frac{x}{n}\mathrm{i}\right)}\right)\,, \tag{4.62a}$$

$$\mathcal{K}^{(a,b)}_{[n]}(x) = \mathcal{K}^{(b,a)}_{[n]}(x) = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{j=0}^{a-1} \ln \frac{\Gamma\left(\frac{2j+b-a}{2n} + \frac{x}{n}\mathrm{i}\right)\Gamma\left(1 + \frac{2j-b-a}{2n} - \frac{x}{n}\mathrm{i}\right)}{\Gamma\left(\frac{2j+b-a}{2n} - \frac{x}{n}\mathrm{i}\right)\Gamma\left(1 + \frac{2j-b-a}{2n} + \frac{x}{n}\mathrm{i}\right)}\right)\,, \tag{4.62b}$$

with the restriction $a < b$. The integration constants are again obtained by considering the set of NLIEs (4.55) for $x \to \infty$. We find

$$c^{(1)}_1 = (-3\mu_1 + \mu_2 + \mu_3 + \mu_4)/4\,, \qquad c^{(1)}_2 = (\mu_1 - 3\mu_2 + \mu_3 + \mu_4)/4\,, \tag{4.63a}$$

$$c^{(1)}_3 = (\mu_1 + \mu_2 - 3\mu_3 + \mu_4)/4\,, \qquad c^{(1)}_4 = (\mu_1 + \mu_2 + \mu_3 - 3\mu_4)/4\,, \tag{4.63b}$$

$$c^{(2)}_1 = (-\mu_1 - \mu_2 + \mu_3 + \mu_4)/2\,, \qquad c^{(2)}_2 = (-\mu_1 + \mu_2 - \mu_3 + \mu_4)/2\,, \tag{4.63c}$$

$$c^{(2)}_3 = (-\mu_1 + \mu_2 + \mu_3 - \mu_4)/2\,, \qquad c^{(2)}_4 = (\mu_1 - \mu_2 - \mu_3 + \mu_4)/2\,, \tag{4.63d}$$

$$c^{(2)}_5 = (\mu_1 - \mu_2 + \mu_3 - \mu_4)/2\,, \qquad c^{(2)}_6 = (\mu_1 + \mu_2 - \mu_3 - \mu_4)/2\,, \tag{4.63e}$$

$$c^{(3)}_1 = (-\mu_1 - \mu_2 - \mu_3 + 3\mu_4)/4\,, \qquad c^{(3)}_2 = (-\mu_1 - \mu_2 + 3\mu_3 - \mu_4)/4\,, \tag{4.63f}$$

$$c^{(3)}_3 = (-\mu_1 + 3\mu_2 - \mu_3 - \mu_4)/4\,, \qquad c^{(3)}_4 = (3\mu_1 - \mu_2 - \mu_3 - \mu_4)/4\,. \tag{4.63g}$$

Therewith, the self-consistent set of coupled NLIEs for the $sl(4)$-symmetric case of the Uimin-Sutherland model is complete.

Due to the connection of the auxiliary functions $B^{(a)}_{1,j}(x)$ to the auxiliary functions $Y^{(a)}_1(x)$ of the TBA approach via equation (4.48), we may exploit the expression (3.33) for $n = 4$ to finally regain the largest eigenvalue $\Lambda^{(1)}_1(x)$ of the QTM. Alternatively, the same reasoning as in the $sl(2)$-symmetric case may be applied, since also here the eigenvalue function already appeared in the derivation of the NLIEs. Either way, we find the final expression

$$\ln \Lambda^{(1)}_1(0) = -\beta\left\{J\left(1 - \frac{\pi}{4} - \frac{3}{2}\ln 2\right) - \frac{1}{4}\sum_{j=1}^{4}\mu_j\right\} + \sum_{a=1}^{4}\sum_{j=1}^{d_a}\left[V^{(a)}_{[4]} * \ln B^{(a)}_j\right](0)\,. \tag{4.64}$$

We like to note that the functions $V^{(a)}_{[n]}(x)$ and the transcendental kernel functions $\mathcal{K}^{(a,b)}_{[n]}(x)$ used in this section have already been defined in a form, which will be valid for the general $sl(n)$-symmetric case. For example, it can be checked that the functions $V^{(a)}(x)$ and $\mathcal{K}^{(a,b)}(x)$ that appeared in the $sl(2)$- and $sl(3)$-symmetric cases are of the same form as given by equations (4.56) and (4.62) for $n = 2$ and 3. The general form of the functions $V_{[n]}^{(a)}(x) = A_{[n]}^{(1,a)}(x)$ follows from the TBA approach, see equations (3.26) and (3.33). The general form of the transcendental kernels follows from their connection to the S matrix of elementary excitations, which for the general $sl(n)$-symmetric model has first been derived by Kulish and Reshetikhin [50]. The relation is

$$\mathcal{K}_{[n]}^{(a,b)}(x) = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln S_{[n]}^{(a,b)}(x)\right) = \frac{\mathrm{d}}{\mathrm{d}x}\varphi_{[n]}^{(a,b)}(x)\,, \tag{4.65}$$

where $S_{[n]}^{(a,b)}(x)$ is the S matrix element for the scattering of particles of type a and b, and $\varphi_{[n]}^{(a,b)}(x)$ is the corresponding phase shift. Note that the functions $\mathcal{K}_{[n]}^{(a,b)}(x)$ also appear as kernel functions in the zero-temperature limit of the TBA equations (3.43).

4.2.2 The $sl(3|1)$-symmetric case

Although there are so far no interesting physical applications of the $sl(3|1)$-symmetric case of the Uimin-Sutherland model, we will nevertheless treat this case for completeness. It will also be the basis for the conjecture of the NLIEs for the $sl(4|1)$-symmetric case in the next chapter. Depending on the explicit choice of grading for the basis states, there are four equivalent formulations. We will choose the grading $(+-++)$ for the following calculations. We find a total of seven auxiliary functions, half as many as for the ungraded $sl(4)$-symmetric case,

$$b_{1,1}^{(1)}(x) = \left.\frac{\boxed{1}}{\boxed{2}+\boxed{3}+\boxed{4}}\right|_{x+\mathrm{i}/2}, \tag{4.66a}$$

$$b_{1,2}^{(1)}(x) = \left.\frac{\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\right)}{\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)}\right|_{x}, \tag{4.66b}$$

$$b_{1,3}^{(1)}(x) = \left.\frac{\boxed{4}}{\boxed{1}+\boxed{2}+\boxed{3}}\right|_{x-\mathrm{i}/2}, \tag{4.66c}$$

$$b_{1,1}^{(2)}(x) = \left.\frac{\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\right)}{\left(\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)\cdot\left(\begin{array}{|c|}\hline 2\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)}\right|_{x+\mathrm{i}/2}, \tag{4.66d}$$

$$b_{1,2}^{(2)}(x) = \left.\frac{\boxed{1}\cdot\boxed{4}}{\left(\boxed{2}+\boxed{3}\right)\cdot\left(\boxed{1}+\boxed{2}+\boxed{3}+\boxed{4}\right)}\right|_{x}, \tag{4.66e}$$

$$b_{1,3}^{(2)}(x) = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}\cdot\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}}{\left(\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}\right)\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}\right)}\right|_{x-\mathrm{i}/2}, \tag{4.66f}$$

$$b_{1,1}^{(3)}(x) = \left.\frac{\substack{\boxed{1}\\\boxed{3}} \cdot \substack{\boxed{3}\\\boxed{4}}}{\substack{\boxed{2}\\\boxed{3}} \cdot \left(\substack{\boxed{1}\\\boxed{2}} + \substack{\boxed{1}\\\boxed{3}} + \substack{\boxed{1}\\\boxed{4}} + \substack{\boxed{2}\\\boxed{2}} + \substack{\boxed{2}\\\boxed{3}} + \substack{\boxed{2}\\\boxed{4}} + \substack{\boxed{3}\\\boxed{4}}\right)}\right|_x . \tag{4.66g}$$

As usual, we define the uppercase auxiliary functions $B_{1,j}^{(a)}(x) = b_{1,j}^{(a)}(x) + 1$. In terms of the Young tableaux, we find the explicit form

$$B_{1,1}^{(1)}(x) = \left.\frac{\boxed{1} + \boxed{2} + \boxed{3} + \boxed{4}}{\boxed{2} + \boxed{3} + \boxed{4}}\right|_{x+\mathrm{i}/2} , \tag{4.67a}$$

$$B_{1,2}^{(1)}(x) = \left.\frac{\left(\substack{\boxed{1}\\\boxed{4}} + \substack{\boxed{2}\\\boxed{4}} + \substack{\boxed{3}\\\boxed{4}}\right) \cdot \left(\substack{\boxed{1}\\\boxed{2}} + \substack{\boxed{1}\\\boxed{3}} + \substack{\boxed{1}\\\boxed{4}}\right)}{\substack{\boxed{1}\\\boxed{4}} \cdot \left(\substack{\boxed{1}\\\boxed{2}} + \substack{\boxed{1}\\\boxed{3}} + \substack{\boxed{1}\\\boxed{4}} + \substack{\boxed{2}\\\boxed{2}} + \substack{\boxed{2}\\\boxed{3}} + \substack{\boxed{2}\\\boxed{4}} + \substack{\boxed{3}\\\boxed{4}}\right)}\right|_x , \tag{4.67b}$$

$$B_{1,3}^{(1)}(x) = \left.\frac{\boxed{1} + \boxed{2} + \boxed{3} + \boxed{4}}{\boxed{1} + \boxed{2} + \boxed{3}}\right|_{x-\mathrm{i}/2} , \tag{4.67c}$$

$$B_{1,1}^{(2)}(x) = \left.\frac{\left(\substack{\boxed{2}\\\boxed{4}} + \substack{\boxed{3}\\\boxed{4}}\right) \cdot \left(\substack{\boxed{1}\\\boxed{2}} + \substack{\boxed{1}\\\boxed{3}} + \substack{\boxed{1}\\\boxed{4}} + \substack{\boxed{2}\\\boxed{2}} + \substack{\boxed{2}\\\boxed{3}} + \substack{\boxed{2}\\\boxed{4}} + \substack{\boxed{3}\\\boxed{4}}\right)}{\left(\substack{\boxed{1}\\\boxed{4}} + \substack{\boxed{2}\\\boxed{4}} + \substack{\boxed{3}\\\boxed{4}}\right) \cdot \left(\substack{\boxed{2}\\\boxed{2}} + \substack{\boxed{2}\\\boxed{3}} + \substack{\boxed{2}\\\boxed{4}} + \substack{\boxed{3}\\\boxed{4}}\right)}\right|_{x+\mathrm{i}/2} , \tag{4.67d}$$

$$B_{1,2}^{(2)}(x) = \left.\frac{\left(\boxed{1} + \boxed{2} + \boxed{3}\right) \cdot \left(\boxed{2} + \boxed{3} + \boxed{4}\right)}{\left(\boxed{2} + \boxed{3}\right) \cdot \left(\boxed{1} + \boxed{2} + \boxed{3} + \boxed{4}\right)}\right|_x , \tag{4.67e}$$

$$B_{1,3}^{(2)}(x) = \left.\frac{\substack{\boxed{1}\\\boxed{4}} \cdot \left(\substack{\boxed{1}\\\boxed{2}} + \substack{\boxed{1}\\\boxed{3}} + \substack{\boxed{1}\\\boxed{4}} + \substack{\boxed{2}\\\boxed{2}} + \substack{\boxed{2}\\\boxed{3}} + \substack{\boxed{2}\\\boxed{4}} + \substack{\boxed{3}\\\boxed{4}}\right)}{\left(\substack{\boxed{1}\\\boxed{4}} + \substack{\boxed{2}\\\boxed{4}}\right) \cdot \left(\substack{\boxed{1}\\\boxed{2}} + \substack{\boxed{1}\\\boxed{3}} + \substack{\boxed{1}\\\boxed{4}}\right)}\right|_{x-\mathrm{i}/2} , \tag{4.67f}$$

$$B_{1,1}^{(3)}(x) = \left.\frac{\left(\substack{\boxed{1}\\\boxed{3}} + \substack{\boxed{2}\\\boxed{3}}\right) \cdot \left(\substack{\boxed{2}\\\boxed{2}} + \substack{\boxed{2}\\\boxed{3}} + \substack{\boxed{2}\\\boxed{4}} + \substack{\boxed{3}\\\boxed{4}}\right)}{\substack{\boxed{2}\\\boxed{3}} \cdot \left(\substack{\boxed{1}\\\boxed{2}} + \substack{\boxed{1}\\\boxed{3}} + \substack{\boxed{1}\\\boxed{4}} + \substack{\boxed{2}\\\boxed{2}} + \substack{\boxed{2}\\\boxed{3}} + \substack{\boxed{2}\\\boxed{4}} + \substack{\boxed{3}\\\boxed{4}}\right)}\right|_x . \tag{4.67g}$$

The connection of the auxiliary functions to those of the corresponding TBA approach becomes apparent by the relations

$$Y_1^{(a)}(x) = B_{1,1}^{(a)}(x) B_{1,2}^{(a)}(x) B_{1,3}^{(a)}(x) , \qquad Y_1^{(3)}(x) = B_{1,1}^{(3)}(x) , \tag{4.68}$$

where $a = 1, 2$. We see that one of the functions of the TBA approach directly enters the set, while the other relations are similar to the $sl(3)$-symmetric case, see equation (4.18). The structure is also analogous to the $sl(2|1)$-symmetric case, see equation (4.31). Comparing the terms appearing in the auxiliary functions with the eigenvalue functions

$$\Lambda_1^{(1)}(x) = \left.\boxed{1} + \boxed{2} + \boxed{3} + \boxed{4}\right|_x , \tag{4.69a}$$

$$\Lambda_1^{(2)}(x) = \left.\boxed{\begin{smallmatrix}1\\2\end{smallmatrix}} + \boxed{\begin{smallmatrix}1\\3\end{smallmatrix}} + \boxed{\begin{smallmatrix}1\\4\end{smallmatrix}} + \boxed{\begin{smallmatrix}2\\2\end{smallmatrix}} + \boxed{\begin{smallmatrix}2\\3\end{smallmatrix}} + \boxed{\begin{smallmatrix}2\\4\end{smallmatrix}} + \boxed{\begin{smallmatrix}3\\4\end{smallmatrix}}\right\|_x , \tag{4.69b}$$

we again see that only partial sums of the Young tableaux in the eigenvalues contribute. Besides the eigenvalue functions, we find the constituents

$$\left.\boxed{1} + \boxed{2}\right\|_x = \frac{\phi_+(x) q_1(x+\mathrm{i}) q_1^{(h)}(x)}{q_2(x)} \cdot \vartheta_1 , \tag{4.70a}$$

$$\left.\boxed{2} + \boxed{3}\right\|_x = \frac{\phi_-(x)\phi_+(x) q_2(x-\mathrm{i}) q_2^{(h)}(x)}{q_1(x) q_3(x)} \cdot \vartheta_2 , \tag{4.70b}$$

$$\left.\boxed{1} + \boxed{2} + \boxed{3}\right\|_x = \frac{\phi_+(x) X_1^{(1)}(x)}{q_3(x)} \cdot \chi_1^{(1)} , \tag{4.70c}$$

$$\left.\boxed{2} + \boxed{3} + \boxed{4}\right\|_x = \frac{\phi_-(x) X_2^{(1)}(x)}{q_1(x)} \cdot \chi_2^{(1)} , \tag{4.70d}$$

$$\left.\boxed{\begin{smallmatrix}2\\2\end{smallmatrix}} + \boxed{\begin{smallmatrix}2\\3\end{smallmatrix}} + \boxed{\begin{smallmatrix}2\\4\end{smallmatrix}} + \boxed{\begin{smallmatrix}3\\4\end{smallmatrix}}\right\|_x = \frac{\phi_-(x-\frac{\mathrm{i}}{2})\phi_-(x+\frac{\mathrm{i}}{2})\phi_+(x-\frac{\mathrm{i}}{2}) q_2(x-\frac{3}{2}\mathrm{i}) X_1^{(2)}(x)}{q_1(x-\frac{\mathrm{i}}{2})} \cdot \chi_1^{(2)} , \tag{4.70e}$$

where the functions $q_j^{(h)}(x)$ and $X_j^{(a)}(x)$ are polynomials due to the cancellation of potential poles. The roots of $q_j^{(h)}(x)$ are given by the hole solutions of the Bethe ansatz equations. We find that $q_j^{(h)}(x)$ and $X_1^{(2)}(x)$ are of degree $N/2$, while the polynomials $X_j^{(1)}(x)$ are of degree N. Again, we additionally define normalized eigenvalue polynomials $\widetilde{\Lambda}_1^{(b)}(x)$ via $\Lambda_1^{(b)}(x) = n_1^{(b)}(x)\widetilde{\Lambda}_1^{(b)}(x) a_1^{(b)}$. All polynomials have the highest coefficient one, since we have separated the constants

$$\vartheta_1 = \mathrm{e}^{\beta\mu_1} + \mathrm{e}^{\beta\mu_2} , \tag{4.71a}$$

$$\vartheta_2 = \mathrm{e}^{\beta\mu_2} + \mathrm{e}^{\beta\mu_3} , \tag{4.71b}$$

$$\chi_1^{(1)} = \mathrm{e}^{\beta\mu_1} + \mathrm{e}^{\beta\mu_2} + \mathrm{e}^{\beta\mu_3} , \tag{4.71c}$$

$$\chi_2^{(1)} = \mathrm{e}^{\beta\mu_2} + \mathrm{e}^{\beta\mu_3} + \mathrm{e}^{\beta\mu_4} , \tag{4.71d}$$

$$\chi_1^{(2)} = \mathrm{e}^{2\beta\mu_2} + \mathrm{e}^{\beta(\mu_2+\mu_3)} + \mathrm{e}^{\beta(\mu_2+\mu_4)} + \mathrm{e}^{\beta(\mu_3+\mu_4)} , \tag{4.71e}$$

$$a_1^{(1)} = \mathrm{e}^{\beta\mu_1} + \mathrm{e}^{\beta\mu_2} + \mathrm{e}^{\beta\mu_3} + \mathrm{e}^{\beta\mu_4} , \tag{4.71f}$$

$$\begin{aligned} a_1^{(2)} = {} & \mathrm{e}^{\beta(\mu_1+\mu_2)} + \mathrm{e}^{\beta(\mu_1+\mu_3)} + \mathrm{e}^{\beta(\mu_1+\mu_4)} + \mathrm{e}^{2\beta\mu_2} \\ & + \mathrm{e}^{\beta(\mu_2+\mu_3)} + \mathrm{e}^{\beta(\mu_2+\mu_4)} + \mathrm{e}^{\beta(\mu_3+\mu_4)} . \end{aligned} \tag{4.71g}$$

Moreover, we know the root structure of all polynomials from numerical calculations. The roots have imaginary parts close to the values

$$q_j(x)\colon 0 , \qquad \widetilde{\Lambda}_1^{(a)}(x)\colon \pm(a+1)/2 , \tag{4.72a}$$

$$q_1^{(h)}(x)\colon +1 , \qquad q_2^{(h)}(x)\colon -1 , \tag{4.72b}$$

$$X_j^{(1)}(x)\colon \pm 1\,, \qquad\qquad X_1^{(2)}(x)\colon -3/2\,. \tag{4.72c}$$

With this knowledge, the auxiliary functions may easily be written in factorized form. We obtain

$$b_{1,1}^{(1)}(x) = \frac{\phi_-(x-\frac{\mathrm{i}}{2})\phi_+(x+\frac{\mathrm{i}}{2})q_1(x+\frac{3}{2}\mathrm{i})}{\phi_-(x+\frac{\mathrm{i}}{2})X_2^{(1)}(x+\frac{\mathrm{i}}{2})} \cdot \frac{\mathrm{e}^{\beta\mu_1}}{\chi_2^{(1)}}\,, \tag{4.73a}$$

$$b_{1,2}^{(1)}(x) = \frac{\phi_-(x-\frac{\mathrm{i}}{2})\phi_+(x+\frac{\mathrm{i}}{2})q_2(x-\frac{3}{2}\mathrm{i})q_2^{(h)}(x+\frac{\mathrm{i}}{2})}{q_1(x+\frac{\mathrm{i}}{2})q_3(x-\frac{\mathrm{i}}{2})\widetilde{\Lambda}_1^{(2)}(x)} \cdot \frac{\mathrm{e}^{\beta\mu_3}\vartheta_2}{a_1^{(2)}}\,, \tag{4.73b}$$

$$b_{1,3}^{(1)}(x) = \frac{\phi_-(x-\frac{\mathrm{i}}{2})\phi_+(x+\frac{\mathrm{i}}{2})q_3(x-\frac{3}{2}\mathrm{i})}{\phi_+(x-\frac{\mathrm{i}}{2})X_1^{(1)}(x-\frac{\mathrm{i}}{2})} \cdot \frac{\mathrm{e}^{\beta\mu_4}}{\chi_1^{(1)}}\,, \tag{4.73c}$$

$$b_{1,1}^{(2)}(x) = \frac{\phi_-(x-\mathrm{i})\phi_+(x+\mathrm{i})q_2^{(h)}(x+\mathrm{i})}{X_1^{(1)}(x)X_1^{(2)}(x+\frac{\mathrm{i}}{2})} \cdot \frac{\mathrm{e}^{\beta(\mu_1+\mu_3)}\vartheta_2}{\chi_1^{(1)}\chi_1^{(2)}}\,, \tag{4.73d}$$

$$b_{1,2}^{(2)}(x) = \frac{\phi_-(x-\mathrm{i})\phi_+(x+\mathrm{i})q_1(x+\mathrm{i})q_3(x-\mathrm{i})}{q_2(x-\mathrm{i})q_2^{(h)}(x)\widetilde{\Lambda}_1^{(1)}(x)} \cdot \frac{\mathrm{e}^{\beta(\mu_1+\mu_4)}}{\vartheta_2 a_1^{(1)}}\,, \tag{4.73e}$$

$$b_{1,3}^{(2)}(x) = \frac{\phi_-(x-\mathrm{i})\phi_+(x+\mathrm{i})q_2(x-2\mathrm{i})}{q_1^{(h)}(x-\mathrm{i})X_2^{(1)}(x)} \cdot \frac{\mathrm{e}^{\beta(\mu_3+\mu_4)}}{\vartheta_1\chi_2^{(1)}}\,, \tag{4.73f}$$

$$b_{1,1}^{(3)}(x) = \frac{\phi_-(x-\frac{3}{2}\mathrm{i})\phi_+(x+\frac{3}{2}\mathrm{i})}{\widetilde{\Lambda}_1^{(2)}(x)} \cdot \frac{\mathrm{e}^{\beta(\mu_1+\mu_3+\mu_4)}}{\mathrm{e}^{\beta\mu_2}a_1^{(2)}}\,. \tag{4.73g}$$

The uppercase auxiliary functions take the form

$$B_{1,1}^{(1)}(x) = \frac{q_1(x+\frac{\mathrm{i}}{2})\widetilde{\Lambda}_1^{(1)}(x+\frac{\mathrm{i}}{2})}{\phi_-(x+\frac{\mathrm{i}}{2})X_2^{(1)}(x+\frac{\mathrm{i}}{2})} \cdot \frac{a_1^{(1)}}{\chi_2^{(1)}}\,, \tag{4.74a}$$

$$B_{1,2}^{(1)}(x) = \frac{X_1^{(1)}(x-\frac{\mathrm{i}}{2})X_2^{(1)}(x+\frac{\mathrm{i}}{2})}{q_1(x+\frac{\mathrm{i}}{2})q_3(x-\frac{\mathrm{i}}{2})\widetilde{\Lambda}_1^{(2)}(x)} \cdot \frac{\chi_1^{(1)}\chi_2^{(1)}}{a_1^{(2)}}\,, \tag{4.74b}$$

$$B_{1,3}^{(1)}(x) = \frac{q_3(x-\frac{\mathrm{i}}{2})\widetilde{\Lambda}_1^{(1)}(x-\frac{\mathrm{i}}{2})}{\phi_+(x-\frac{\mathrm{i}}{2})X_1^{(1)}(x-\frac{\mathrm{i}}{2})} \cdot \frac{a_1^{(1)}}{\chi_1^{(1)}}\,, \tag{4.74c}$$

$$B_{1,1}^{(2)}(x) = \frac{q_2^{(h)}(x)\widetilde{\Lambda}_1^{(2)}(x+\frac{\mathrm{i}}{2})}{X_1^{(1)}(x)X_1^{(2)}(x+\frac{\mathrm{i}}{2})} \cdot \frac{\vartheta_2 a_1^{(2)}}{\chi_1^{(1)}\chi_1^{(2)}}\,, \tag{4.74d}$$

$$B_{1,2}^{(2)}(x) = \frac{X_1^{(1)}(x)X_2^{(1)}(x)}{q_2(x-\mathrm{i})q_2^{(h)}(x)\widetilde{\Lambda}_1^{(1)}(x)} \cdot \frac{\chi_1^{(1)}\chi_2^{(1)}}{\vartheta_2 a_1^{(1)}}\,, \tag{4.74e}$$

$$B_{1,3}^{(2)}(x) = \frac{q_2(x-\mathrm{i})\widetilde{\Lambda}_1^{(2)}(x-\frac{\mathrm{i}}{2})}{q_1^{(h)}(x-\mathrm{i})X_2^{(1)}(x)} \cdot \frac{a_1^{(2)}}{\vartheta_1\chi_2^{(1)}}\,, \tag{4.74f}$$

$$B_{1,1}^{(3)}(x) = \frac{q_1^{(h)}(x-\frac{\mathrm{i}}{2})X_1^{(2)}(x)}{\widetilde{\Lambda}_1^{(2)}(x)} \cdot \frac{\vartheta_1\chi_1^{(2)}}{\mathrm{e}^{\beta\mu_2}a_1^{(2)}}\,. \tag{4.74g}$$

Now it is easy to check, that all auxiliary functions have the desired ANZC property in some strip around the real axis, and we are therefore allowed to apply the transform (4.4). Although there are a total of ten unknown functions involved besides the auxiliary functions, only seven of them remain in the transformed equations for $k < 0$ and $k > 0$, since their roots appear only either above or below the real axis. This allows for the derivation of the NLIEs along the usual lines. For brevity, we will directly skip to the results.

We arrive at the self-consistent set of coupled NLIEs

$$\ln b_{1,j}^{(a)} = -\beta \left(JV^{(a)}(x) + c_j^{(a)} \right) - \sum_{b=1}^{3} \sum_{k=1}^{d_b} \left[\mathbf{K}_{j,k}^{(a,b)} * \ln B_{1,k}^{(b)} \right](x) \,. \tag{4.75}$$

The number d_b gives the number of auxiliary functions in each set b. The driving terms are

$$V^{(a)}(x) = \frac{4a}{4x^2 + a^2} \,, \tag{4.76}$$

and we obtain the integration constants

$$c_1^{(1)} = \mu_2 - \mu_1 \,, \qquad c_2^{(1)} = \mu_2 - \mu_3 \,, \tag{4.77a}$$

$$c_3^{(1)} = \mu_2 - \mu_4 \,, \qquad c_1^{(2)} = 2\mu_2 - \mu_1 - \mu_3 \,, \tag{4.77b}$$

$$c_2^{(2)} = 2\mu_2 - \mu_1 - \mu_4 \,, \qquad c_3^{(2)} = 2\mu_2 - \mu_3 - \mu_4 \,, \tag{4.77c}$$

$$c_1^{(3)} = 3\mu_2 - \mu_1 - \mu_3 - \mu_4 \,. \tag{4.77d}$$

For the complete kernel matrix, we find

$$\begin{aligned}
\mathbf{K}(x) &= \begin{pmatrix} \mathbf{K}^{(1,1)}(x) & \mathbf{K}^{(1,2)}(x) & \mathbf{K}^{(1,3)}(x) \\ \mathbf{K}^{(2,1)}(x) & \mathbf{K}^{(2,2)}(x) & \mathbf{K}^{(2,3)}(x) \\ \mathbf{K}^{(3,1)}(x) & \mathbf{K}^{(3,2)}(x) & \mathbf{K}^{(3,3)}(x) \end{pmatrix} \\
&= \left(\begin{array}{ccc|ccc|c}
0 & K_1(x) & K_1(x) & K_3(x) & K_3(x) & K_4(x) & K_{10}(x) \\
K_2(x) & 0 & K_1(x) & K_3(x) & K_6(x) & K_3(x) & K_{10}(x) \\
K_2(x) & K_2(x) & 0 & K_5(x) & K_3(x) & K_3(x) & K_{10}(x) \\ \hline
K_3(x) & K_3(x) & K_4(x) & K_7(x) & K_8(x) & K_8(x) & K_{11}(x) \\
K_3(x) & K_6(x) & K_3(x) & K_9(x) & K_7(x) & K_8(x) & K_{11}(x) \\
K_5(x) & K_3(x) & K_3(x) & K_9(x) & K_9(x) & K_7(x) & K_{11}(x) \\ \hline
K_{10}(x) & K_{10}(x) & K_{10}(x) & K_{11}(x) & K_{11}(x) & K_{11}(x) & K_{12}(x)
\end{array} \right) .
\end{aligned} \tag{4.78}$$

The explicit kernel functions are

$$K_1(x) = \mathcal{K}^{(1,1)}(x) + \frac{\mathrm{i}}{x + \mathrm{i}} - \frac{\mathrm{i}}{x - 0\mathrm{i}} \,, \qquad K_2(x) = \mathcal{K}^{(1,1)}(x) + \frac{\mathrm{i}}{x + 0\mathrm{i}} - \frac{\mathrm{i}}{x - \mathrm{i}} \,, \tag{4.79a}$$

$$K_3(x) = \mathcal{K}^{(1,2)}(x) \,, \qquad K_4(x) = \mathcal{K}^{(1,2)}(x) + \frac{\mathrm{i}}{x + \frac{3}{2}\mathrm{i}} - \frac{\mathrm{i}}{x + \frac{\mathrm{i}}{2}} \,, \tag{4.79b}$$

$$K_5(x) = \mathcal{K}^{(1,2)}(x) + \frac{\mathrm{i}}{x - \frac{\mathrm{i}}{2}} - \frac{\mathrm{i}}{x - \frac{3}{2}\mathrm{i}} \,, \qquad K_6(x) = \mathcal{K}^{(1,2)}(x) + \frac{\mathrm{i}}{x + \frac{\mathrm{i}}{2}} - \frac{\mathrm{i}}{x - \frac{\mathrm{i}}{2}} \,, \tag{4.79c}$$

$$K_7(x) = \mathcal{K}^{(2,2)}(x)\,, \qquad K_8(x) = \mathcal{K}^{(2,2)}(x) + \frac{\mathrm{i}}{x+\mathrm{i}} - \frac{\mathrm{i}}{x-0\mathrm{i}}\,, \tag{4.79d}$$

$$K_9(x) = \mathcal{K}^{(2,2)}(x) + \frac{\mathrm{i}}{x+0\mathrm{i}} - \frac{\mathrm{i}}{x-\mathrm{i}}\,, \qquad K_{10}(x) = \mathcal{K}^{(1,3)}(x)\,, \tag{4.79e}$$

$$K_{11}(x) = \mathcal{K}^{(2,3)}(x)\,, \qquad K_{12}(x) = \mathcal{K}^{(3,3)}(x)\,, \tag{4.79f}$$

where the common functions have the form

$$\mathcal{K}^{(a,b)}(x) = \int_{-\infty}^{\infty} \left\{ \mathrm{e}^{-(\max(a,b)-1)|k|/2} \frac{\sinh(\min(a,b)k/2)}{\sinh(k/2)} - \delta_{a,b} \right\} \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}k\,. \tag{4.80}$$

Note that these functions have already appeared as the kernel functions of the zero-temperature limit of the TBA equations (3.46).

It is a remarkable fact, that the complete kernel matrix $\mathbf{K}(x)$ looks very similar to the one we have obtained for the $sl(3)$-symmetric case. Despite the additional auxiliary function $b_1^{(3)}(x)$ and the different common functions $\mathcal{K}^{(a,b)}(x)$, the block structure of the matrix and all the rational parts of the kernel functions are exactly the same. Note that this observation also conforms with what we have already found for the $sl(2|1)$-symmetric case.

The expression for the largest eigenvalue of the QTM finally is

$$\ln \Lambda_1^{(1)}(0) = \beta\,(J + \mu_2) + \sum_{a=1}^{3} \sum_{j=1}^{d_a} \left[V^{(a)} * \ln B_{1,j}^{(a)} \right](0)\,. \tag{4.81}$$

4.2.3 The $sl(2|2)$-symmetric case

The structure for the $sl(2|2)$-symmetric case of the Uimin-Sutherland model is a bit more complicated. There are six equivalent possible choices to apply the grading to the basis states. In the following, we will stick to the grading $(+--+)$. We find a total of six auxiliary functions,

$$b_{1,1}^{(1)}(x) = \left. \frac{\boxed{1}\boxed{1} + \boxed{1}\boxed{2} + \boxed{1}\boxed{3} + \boxed{1}\boxed{4} + \boxed{2}\boxed{3} + \boxed{2}\boxed{4} + \boxed{3}\boxed{4} + \boxed{4}\boxed{4}}{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 2\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 3\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}} \right|_{x}\,, \tag{4.82a}$$

$$b_{2,1}^{(1)}(x) = \left. \frac{\boxed{1}\boxed{1} + \boxed{1}\boxed{2} + \boxed{1}\boxed{3} + \boxed{1}\boxed{4}}{\boxed{2}\boxed{3} + \boxed{2}\boxed{4} + \boxed{3}\boxed{4} + \boxed{4}\boxed{4}} \right|_{x+\mathrm{i}/2}\,, \tag{4.82b}$$

$$b_{2,2}^{(1)}(x) = \left. \frac{\boxed{1}\boxed{4} + \boxed{2}\boxed{4} + \boxed{3}\boxed{4} + \boxed{4}\boxed{4}}{\boxed{1}\boxed{1} + \boxed{1}\boxed{2} + \boxed{1}\boxed{3} + \boxed{2}\boxed{3}} \right|_{x-\mathrm{i}/2}\,, \tag{4.82c}$$

$$b_{1,1}^{(2)}(x) = \left. \frac{\boxed{1} + \boxed{2} + \boxed{3} + \boxed{4}}{\boxed{1} + \boxed{2}} \right|_{x} \cdot \left. \frac{\begin{array}{|c|}\hline 3\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}}{\boxed{2}\boxed{3} + \boxed{2}\boxed{4} + \boxed{3}\boxed{4} + \boxed{4}\boxed{4}} \right|_{x+\mathrm{i}/2}\,, \tag{4.82d}$$

$$b^{(2)}_{1,2}(x) = \left.\frac{\boxed{1}+\boxed{2}+\boxed{3}+\boxed{4}}{\boxed{3}+\boxed{4}}\right|_x \cdot \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 2\\\hline\end{array}}{\boxed{1}\boxed{1}+\boxed{1}\boxed{2}+\boxed{1}\boxed{3}+\boxed{2}\boxed{3}}\right|_{x-\mathrm{i}/2}, \tag{4.82e}$$

$$b^{(2)}_{2,1}(x) = \left.\frac{\boxed{2}\boxed{3}\cdot\left(\boxed{1}\boxed{1}+\boxed{1}\boxed{2}+\boxed{1}\boxed{3}+\boxed{1}\boxed{4}+\boxed{2}\boxed{3}+\boxed{2}\boxed{4}+\boxed{3}\boxed{4}+\boxed{4}\boxed{4}\right)}{\boxed{1}\boxed{4}\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)}\right|_x. \tag{4.82f}$$

By adding one, we arrive at the uppercase auxiliary functions $B^{(a)}_{m,j}(x)$. These may be written in an analogous way,

$$B^{(1)}_{1,1}(x) = \frac{\left(\boxed{1}+\boxed{2}+\boxed{3}+\boxed{4}\right)\big|_{x-\mathrm{i}/2}\cdot\left(\boxed{1}+\boxed{2}+\boxed{3}+\boxed{4}\right)\big|_{x+\mathrm{i}/2}}{\left.\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right|_x}, \tag{4.83a}$$

$$B^{(1)}_{2,1}(x) = \left.\frac{\boxed{1}\boxed{1}+\boxed{1}\boxed{2}+\boxed{1}\boxed{3}+\boxed{1}\boxed{4}+\boxed{2}\boxed{3}+\boxed{2}\boxed{4}+\boxed{3}\boxed{4}+\boxed{4}\boxed{4}}{\boxed{2}\boxed{3}+\boxed{2}\boxed{4}+\boxed{3}\boxed{4}+\boxed{4}\boxed{4}}\right|_{x+\mathrm{i}/2}, \tag{4.83b}$$

$$B^{(1)}_{2,2}(x) = \left.\frac{\boxed{1}\boxed{1}+\boxed{1}\boxed{2}+\boxed{1}\boxed{3}+\boxed{2}\boxed{3}+\boxed{1}\boxed{4}+\boxed{2}\boxed{4}+\boxed{3}\boxed{4}+\boxed{4}\boxed{4}}{\boxed{1}\boxed{1}+\boxed{1}\boxed{2}+\boxed{1}\boxed{3}+\boxed{2}\boxed{3}}\right|_{x-\mathrm{i}/2}, \tag{4.83c}$$

$$B^{(2)}_{1,1}(x) = \left.\frac{\boxed{3}+\boxed{4}}{\boxed{1}+\boxed{2}}\right|_x \cdot \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}}{\boxed{2}\boxed{3}+\boxed{2}\boxed{4}+\boxed{3}\boxed{4}+\boxed{4}\boxed{4}}\right|_{x+\mathrm{i}/2}, \tag{4.83d}$$

$$B^{(2)}_{1,2}(x) = \left.\frac{\boxed{1}+\boxed{2}}{\boxed{3}+\boxed{4}}\right|_x \cdot \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}}{\boxed{1}\boxed{1}+\boxed{1}\boxed{2}+\boxed{1}\boxed{3}+\boxed{2}\boxed{3}}\right|_{x-\mathrm{i}/2}, \tag{4.83e}$$

$$B^{(2)}_{2,1}(x) = \left.\frac{\left(\boxed{1}\boxed{1}+\boxed{1}\boxed{2}+\boxed{1}\boxed{3}+\boxed{2}\boxed{3}\right)\cdot\left(\boxed{2}\boxed{3}+\boxed{2}\boxed{4}+\boxed{3}\boxed{4}+\boxed{4}\boxed{4}\right)}{\boxed{1}\boxed{4}\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)}\right|_x. \tag{4.83f}$$

We find the connection to the auxiliary functions of the TBA approach

$$Y^{(1)}_1(x) = B^{(1)}_{1,1}(x)\,, \qquad Y^{(1)}_2(x) = B^{(1)}_{2,1}(x)B^{(1)}_{2,2}(x)\,, \tag{4.84a}$$

$$\overline{Y}^{(2)}_1(x) = B^{(2)}_{1,1}(x)B^{(2)}_{1,2}(x)\,, \qquad \overline{Y}^{(2)}_2(x) = B^{(2)}_{2,1}(x)\,, \tag{4.84b}$$

which is structurally a bit different from the previous cases, since the functions $\overline{Y}^{(2)}_j(x)$ appear instead of $Y^{(2)}_j(x)$.

In analogy to the previous cases, the sums of Young tableaux appearing in the auxiliary functions are partial sums of the tableaux in the eigenvalue functions

$$\Lambda_1^{(1)}(x) = \boxed{1} + \boxed{2} + \boxed{3} + \boxed{4}\Big|_x \,, \tag{4.85a}$$

$$\Lambda_2^{(1)}(x) = \boxed{1\,|\,1} + \boxed{1\,|\,2} + \boxed{1\,|\,3} + \boxed{1\,|\,4} + \boxed{2\,|\,3} + \boxed{2\,|\,4} + \boxed{3\,|\,4} + \boxed{4\,|\,4}\Big|_x \,, \tag{4.85b}$$

$$\Lambda_1^{(2)}(x) = \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 2\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 3\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\Bigg|_x \,. \tag{4.85c}$$

This time, also the fused eigenvalue function $\Lambda_2^{(1)}(x)$ is important. As usual, we introduce the normalized eigenvalue polynomials $\widetilde{\Lambda}_m^{(b)}(x)$, where $\Lambda_m^{(b)}(x) = n_m^{(b)}(x)\widetilde{\Lambda}_m^{(b)}(x)a_m^{(b)}$. The constants $a_m^{(b)}$ are

$$a_1^{(1)} = \mathrm{e}^{\beta\mu_1} + \mathrm{e}^{\beta\mu_2} + \mathrm{e}^{\beta\mu_3} + \mathrm{e}^{\beta\mu_4} \,, \tag{4.86a}$$

$$\begin{aligned} a_2^{(1)} &= \mathrm{e}^{2\beta\mu_1} + \mathrm{e}^{\beta(\mu_1+\mu_2)} + \mathrm{e}^{\beta(\mu_1+\mu_3)} + \mathrm{e}^{\beta(\mu_1+\mu_4)} \\ &\quad + \mathrm{e}^{\beta(\mu_2+\mu_3)} + \mathrm{e}^{\beta(\mu_2+\mu_4)} + \mathrm{e}^{\beta(\mu_3+\mu_4)} + \mathrm{e}^{2\beta\mu_4} \,, \end{aligned} \tag{4.86b}$$

$$\begin{aligned} a_1^{(2)} &= \mathrm{e}^{\beta(\mu_1+\mu_2)} + \mathrm{e}^{\beta(\mu_1+\mu_3)} + \mathrm{e}^{\beta(\mu_1+\mu_4)} + \mathrm{e}^{2\beta\mu_2} \\ &\quad + \mathrm{e}^{\beta(\mu_2+\mu_3)} + \mathrm{e}^{\beta(\mu_2+\mu_4)} + \mathrm{e}^{2\beta\mu_3} + \mathrm{e}^{\beta(\mu_3+\mu_4)} \,. \end{aligned} \tag{4.86c}$$

In addition to the eigenvalue functions themselves, we identify the terms

$$\boxed{1} + \boxed{2}\Big|_x = \frac{\phi_+(x)q_1(x+\mathrm{i})q_1^{(h)}(x)}{q_2(x)} \cdot \vartheta_1 \,, \tag{4.87a}$$

$$\boxed{3} + \boxed{4}\Big|_x = \frac{\phi_-(x)q_3(x-\mathrm{i})q_3^{(h)}(x)}{q_2(x)} \cdot \vartheta_3 \,, \tag{4.87b}$$

$$\boxed{1\,|\,1} + \boxed{1\,|\,2} + \boxed{1\,|\,3} + \boxed{2\,|\,3}\Big|_x = \frac{\phi_-(x-\frac{\mathrm{i}}{2})\phi_+(x-\frac{\mathrm{i}}{2})\phi_+(x+\frac{\mathrm{i}}{2})q_1(x+\frac{3}{2}\mathrm{i})X_{2,1}^{(1)}(x)}{q_3(x-\frac{\mathrm{i}}{2})} \cdot \chi_{2,1}^{(1)} \,, \tag{4.87c}$$

$$\boxed{2\,|\,3} + \boxed{2\,|\,4} + \boxed{3\,|\,4} + \boxed{4\,|\,4}\Big|_x = \frac{\phi_-(x-\frac{\mathrm{i}}{2})\phi_-(x+\frac{\mathrm{i}}{2})\phi_+(x+\frac{\mathrm{i}}{2})q_3(x-\frac{3}{2}\mathrm{i})X_{2,2}^{(1)}(x)}{q_1(x+\frac{\mathrm{i}}{2})} \cdot \chi_{2,2}^{(1)} \,, \tag{4.87d}$$

where the separated constants are

$$\vartheta_1 = \mathrm{e}^{\beta\mu_1} + \mathrm{e}^{\beta\mu_2} \,, \qquad \vartheta_2 = \mathrm{e}^{\beta\mu_3} + \mathrm{e}^{\beta\mu_4} \,, \tag{4.88a}$$

$$\chi_{2,1}^{(1)} = (\mathrm{e}^{\beta\mu_1} + \mathrm{e}^{\beta\mu_2})(\mathrm{e}^{\beta\mu_1} + \mathrm{e}^{\beta\mu_3}) \,, \qquad \chi_{2,2}^{(1)} = (\mathrm{e}^{\beta\mu_2} + \mathrm{e}^{\beta\mu_4})(\mathrm{e}^{\beta\mu_3} + \mathrm{e}^{\beta\mu_4}) \,. \tag{4.88b}$$

Thus, the functions $q_j^{(h)}(x)$ and $X_{2,j}^{(1)}(x)$ are each polynomials of degree $N/2$ with one as highest coefficient. Like before, the roots of the polynomials $q_j^{(h)}(x)$ are the hole solutions of the corresponding Bethe ansatz equations. Numerically, the roots of the polynomials appearing in the auxiliary functions for finite N are found to have imaginary parts close to the values

$$q_j(x)\colon 0\,, \qquad \widetilde{\Lambda}_m^{(a)}(x)\colon \pm(a+m)/2\,, \tag{4.89a}$$

$$q_1^{(h)}(x)\colon +1\,, \qquad q_3^{(h)}(x)\colon -1\,, \tag{4.89b}$$

$$X_{2,1}^{(1)}(x)\colon +3/2\,, \qquad X_{2,2}^{(1)}(x)\colon -3/2\,. \tag{4.89c}$$

Using this information, we can easily verify all auxiliary functions to have the ANZC property in some strip $-1/2 \lesssim \Im(x) \lesssim 1/2$, which is necessary for the following calculation, and pose the functions in the factorized form

$$b_{1,1}^{(1)}(x) = \frac{\phi_-(x-\frac{\mathrm{i}}{2})\phi_+(x+\frac{\mathrm{i}}{2})\widetilde{\Lambda}_2^{(1)}(x)}{\phi_-(x+\frac{\mathrm{i}}{2})\phi_+(x-\frac{\mathrm{i}}{2})\widetilde{\Lambda}_1^{(2)}(x)} \cdot \frac{a_2^{(1)}}{a_1^{(2)}}\,, \tag{4.90a}$$

$$b_{2,1}^{(1)}(x) = \frac{q_1(x+2\mathrm{i})\widetilde{\Lambda}_1^{(1)}(x)}{\phi_-(x+\mathrm{i})q_3(x-\mathrm{i})X_{2,2}^{(1)}(x+\frac{\mathrm{i}}{2})} \cdot \frac{\mathrm{e}^{\beta\mu_1}a_1^{(1)}}{\chi_{2,2}^{(1)}}\,, \tag{4.90b}$$

$$b_{2,2}^{(1)}(x) = \frac{q_3(x-2\mathrm{i})\widetilde{\Lambda}_1^{(1)}(x)}{\phi_+(x-\mathrm{i})q_1(x+\mathrm{i})X_{2,1}^{(1)}(x-\frac{\mathrm{i}}{2})} \cdot \frac{\mathrm{e}^{\beta\mu_4}a_1^{(1)}}{\chi_{2,1}^{(1)}}\,, \tag{4.90c}$$

$$b_{1,1}^{(2)}(x) = \frac{q_3^{(h)}(x+\mathrm{i})\widetilde{\Lambda}_1^{(1)}(x)}{\phi_+(x+\mathrm{i})q_1^{(h)}(x)X_{2,2}^{(1)}(x+\frac{\mathrm{i}}{2})} \cdot \frac{\mathrm{e}^{\beta\mu_3}\vartheta_3 a_1^{(1)}}{\vartheta_1\chi_{2,2}^{(1)}}\,, \tag{4.90d}$$

$$b_{1,2}^{(2)}(x) = \frac{q_1^{(h)}(x-\mathrm{i})\widetilde{\Lambda}_1^{(1)}(x)}{\phi_-(x-\mathrm{i})q_3^{(h)}(x)X_{2,1}^{(1)}(x-\frac{\mathrm{i}}{2})} \cdot \frac{\mathrm{e}^{\beta\mu_2}\vartheta_1 a_1^{(1)}}{\vartheta_3\chi_{2,1}^{(1)}}\,, \tag{4.90e}$$

$$b_{2,1}^{(2)}(x) = \frac{\widetilde{\Lambda}_2^{(1)}(x)}{\widetilde{\Lambda}_1^{(2)}(x)} \cdot \frac{\mathrm{e}^{\beta(\mu_2+\mu_3)}a_2^{(1)}}{\mathrm{e}^{\beta(\mu_1+\mu_4)}a_1^{(2)}}\,. \tag{4.90f}$$

The uppercase auxiliary functions are analogously written in the form

$$B_{1,1}^{(1)}(x) = \frac{\widetilde{\Lambda}_1^{(1)}(x-\frac{\mathrm{i}}{2})\widetilde{\Lambda}_1^{(1)}(x+\frac{\mathrm{i}}{2})}{\phi_-(x+\frac{\mathrm{i}}{2})\phi_+(x-\frac{\mathrm{i}}{2})\widetilde{\Lambda}_1^{(2)}(x)} \cdot \frac{(a_1^{(1)})^2}{a_1^{(2)}}\,, \tag{4.91a}$$

$$B_{2,1}^{(1)}(x) = \frac{q_1(x+\mathrm{i})\widetilde{\Lambda}_2^{(1)}(x+\frac{\mathrm{i}}{2})}{\phi_-(x+\mathrm{i})q_3(x-\mathrm{i})X_{2,2}^{(1)}(x+\frac{\mathrm{i}}{2})} \cdot \frac{a_2^{(1)}}{\chi_{2,2}^{(1)}}\,, \tag{4.91b}$$

$$B_{2,2}^{(1)}(x) = \frac{q_3(x-\mathrm{i})\widetilde{\Lambda}_2^{(1)}(x-\frac{\mathrm{i}}{2})}{\phi_+(x-\mathrm{i})q_1(x+\mathrm{i})X_{2,1}^{(1)}(x-\frac{\mathrm{i}}{2})} \cdot \frac{a_2^{(1)}}{\chi_{2,1}^{(1)}}\,, \tag{4.91c}$$

$$B_{1,1}^{(2)}(x) = \frac{q_3^{(h)}(x)\widetilde{\Lambda}_1^{(2)}(x+\frac{\mathrm{i}}{2})}{\phi_+(x+\mathrm{i})q_1^{(h)}(x)X_{2,2}^{(1)}(x+\frac{\mathrm{i}}{2})} \cdot \frac{\vartheta_3 a_1^{(2)}}{\vartheta_1\chi_{2,2}^{(1)}}\,, \tag{4.91d}$$

$$B_{1,2}^{(2)}(x) = \frac{q_1^{(h)}(x)\widetilde{\Lambda}_1^{(2)}(x-\frac{\mathrm{i}}{2})}{\phi_-(x-\mathrm{i})q_3^{(h)}(x)X_{2,1}^{(1)}(x-\frac{\mathrm{i}}{2})} \cdot \frac{\vartheta_1 a_1^{(2)}}{\vartheta_3\chi_{2,1}^{(1)}}\,, \tag{4.91e}$$

$$B_{2,1}^{(2)}(x) = \frac{X_{2,1}^{(1)}(x) X_{2,2}^{(1)}(x)}{\widetilde{\Lambda}_1^{(2)}(x)} \cdot \frac{\chi_{2,1}^{(1)} \chi_{2,2}^{(1)}}{\mathrm{e}^{\beta(\mu_1+\mu_4)} a_1^{(2)}} \,. \tag{4.91f}$$

We observe exactly nine unknown functions on the right hand sides of the equations. Some of the singularities of the auxiliary functions caused by these unknowns are, however, located only above or below the real axis. The application of the Fourier transform to the logarithmic derivative of the equations, see (4.4), thus reduces the number of unknown functions in each of the subcases $k < 0$ and $k > 0$ to six. As a consequence, all unknown functions can be expressed solely in terms of the uppercase auxiliary functions $B_{m,j}^{(a)}(x)$.

Substituting the results back into the transforms of the functions $b_{m,j}^{(a)}(x)$ and afterwards applying the inverse transform like in the previous derivations, we are again led to a coupled set of NLIEs. It is of the well-known structure

$$\ln b_{m,j}^{(a)}(x) = -\beta \left(J V_m^{(a)}(x) + c_{m,j}^{(a)} \right) - \sum_{b=1}^{2} \sum_{l=1}^{2} \sum_{k=1}^{d_{b,l}} \left[\mathbf{K}_{m,l;j,k}^{(a,b)} * \ln B_{l,k}^{(b)} \right](x) \,. \tag{4.92}$$

Here, $d_{b,l}$ gives the number of auxiliary functions in the set specified by the fusion levels b and l. We find the driving terms to be

$$V_1^{(1)}(x) = \frac{\pi}{\cosh(\pi x)} \,, \qquad V_2^{(1)}(x) = 0 \,, \tag{4.93a}$$

$$V_1^{(2)}(x) = \mathrm{i} \frac{\mathrm{d}}{\mathrm{d}x} \left(\ln \frac{\Gamma(1 + \frac{x}{2}\mathrm{i})\Gamma(\frac{1}{2} - \frac{x}{2}\mathrm{i})}{\Gamma(1 - \frac{x}{2}\mathrm{i})\Gamma(\frac{1}{2} + \frac{x}{2}\mathrm{i})} \right) , \quad V_2^{(2)}(x) = \mathrm{i} \frac{\mathrm{d}}{\mathrm{d}x} \left(\ln \frac{\Gamma(\frac{5}{4} + \frac{x}{2}\mathrm{i})\Gamma(\frac{3}{4} - \frac{x}{2}\mathrm{i})}{\Gamma(\frac{5}{4} - \frac{x}{2}\mathrm{i})\Gamma(\frac{3}{2} + \frac{x}{2}\mathrm{i})} \right) , \tag{4.93b}$$

and the integration constants are

$$c_{1,1}^{(1)} = 0 \,, \qquad c_{2,1}^{(1)} = (\mu_4 - \mu_1)/2 \,, \tag{4.94a}$$

$$c_{2,2}^{(1)} = (\mu_1 - \mu_4)/2 \,, \qquad c_{1,1}^{(2)} = (\mu_1 + \mu_4 - 2\mu_3)/2 \,, \tag{4.94b}$$

$$c_{1,2}^{(2)} = (\mu_1 + \mu_4 - 2\mu_2)/2 \,, \qquad c_{2,1}^{(2)} = \mu_1 + \mu_4 - \mu_2 - \mu_3 \,. \tag{4.94c}$$

The complete kernel matrix has the structure

$$\mathbf{K}(x) = \begin{pmatrix} \mathbf{K}_{1,1}^{(1,1)}(x) & \mathbf{K}_{1,2}^{(1,1)}(x) & \mathbf{K}_{1,1}^{(1,2)}(x) & \mathbf{K}_{1,2}^{(1,2)}(x) \\ \mathbf{K}_{2,1}^{(1,1)}(x) & \mathbf{K}_{2,2}^{(1,1)}(x) & \mathbf{K}_{2,1}^{(1,2)}(x) & \mathbf{K}_{2,2}^{(1,2)}(x) \\ \mathbf{K}_{1,1}^{(2,1)}(x) & \mathbf{K}_{1,2}^{(2,1)}(x) & \mathbf{K}_{1,1}^{(2,2)}(x) & \mathbf{K}_{1,2}^{(2,2)}(x) \\ \mathbf{K}_{2,1}^{(2,1)}(x) & \mathbf{K}_{2,2}^{(2,1)}(x) & \mathbf{K}_{2,1}^{(2,2)}(x) & \mathbf{K}_{2,2}^{(2,2)}(x) \end{pmatrix}$$

$$= \left(\begin{array}{c|cc|cc|c} 0 & -W(x) & -W(x) & W(x) & W(x) & 0 \\ \hline -W(x) & K_0(x) & K_1(x) & 0 & 0 & W(x) \\ -W(x) & K_2(x) & K_0(x) & 0 & 0 & W(x) \\ \hline -W(x) & 0 & 0 & K_0(x) & K_1(x) & W(x) \\ -W(x) & 0 & 0 & K_2(x) & K_0(x) & W(x) \\ \hline 0 & -W(x) & -W(x) & W(x) & W(x) & 0 \end{array}\right) . \tag{4.95}$$

where the kernel $W(x)$ is given by

$$W(x) = \frac{\pi}{\cosh(\pi x)}\,, \tag{4.96}$$

and the kernels $K_j(x)$ are exactly the kernels we already know from the $sl(2)$-symmetric case, confer equation (4.10),

$$K_0(x) = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\frac{\Gamma(1+\frac{x}{2}\mathrm{i})\Gamma(\frac{1}{2}-\frac{x}{2}\mathrm{i})}{\Gamma(1-\frac{x}{2}\mathrm{i})\Gamma(\frac{1}{2}+\frac{x}{2}\mathrm{i})}\right)\,, \tag{4.97a}$$

$$K_1(x) = K_0(x) + \frac{\mathrm{i}}{x+\mathrm{i}} - \frac{\mathrm{i}}{x-0\mathrm{i}}\,, \quad K_2(x) = K_0(x) + \frac{\mathrm{i}}{x+0\mathrm{i}} - \frac{\mathrm{i}}{x-\mathrm{i}}\,. \tag{4.97b}$$

We like to stress that unlike in the previous cases, the kernel matrix $\mathbf{K}(x)$ is no longer Hermitian here. As a consequence, an analytical treatment of the low-temperature behaviour similar to [41] will not be possible in this case.

Once the auxiliary functions are fixed by the NLIEs, the largest eigenvalue of the QTM follows from

$$\begin{aligned}\ln\Lambda_1^{(1)}(0) = -\beta\left\{J(1-2\ln 2) - \frac{\mu_1+\mu_4}{2}\right\} + \left[V_1^{(1)} * \ln B_{1,1}^{(1)}\right](0) \\ - \sum_{m=1}^{2}\sum_{j=1}^{d_{2,m}}\left[V_m^{(2)} * \ln B_{m,j}^{(2)}\right](0)\,.\end{aligned} \tag{4.98}$$

4.3 Limiting cases of the NLIEs

In order to check the validity of the derived NLIEs, it is possible to consider several limiting cases. In the following sections, we will briefly discuss some of these results.

4.3.1 Reducing the degrees of freedom

Considering the Hamiltonian of the Uimin-Sutherland model (2.4) it is immediately obvious, that shifting one of the chemical potentials μ_α to minus infinity effectively freezes out the corresponding basis state. On the level of the coupled set of NLIEs, however, the changes are less trivial. Therefore, it is a good test to check that the NLIEs for the simpler models are contained in the more complicated ones. This method has first been used in [27] to regain the NLIEs for the $sl(2)$- from those for the $sl(3)$-symmetric case.

As an example, we will consider the $sl(4)$-symmetric case here. We choose to freeze out the state $\alpha = 4$ and accordingly treat the limit $\mu_4 \to -\infty$. From the asymptotic behaviour of the auxiliary functions (4.42)–(4.44) we observe

$$b_{1,1}^{(1)}(x) = \mathcal{O}(1)\,, \qquad b_{1,2}^{(1)}(x) = \mathcal{O}(1)\,, \qquad b_{1,3}^{(1)}(x) = \mathcal{O}(1)\,, \tag{4.99a}$$

$$b_{1,4}^{(1)}(x) = \mathcal{O}(\mathrm{e}^{\beta\mu_4})\,, \qquad b_{1,1}^{(2)}(x) = \mathcal{O}(1)\,, \qquad b_{1,2}^{(2)}(x) = \mathcal{O}(1)\,, \tag{4.99b}$$

$$b_{1,3}^{(2)}(x) = \mathcal{O}(\mathrm{e}^{\beta\mu_4}) , \qquad b_{1,4}^{(2)}(x) = \mathcal{O}(1) , \qquad b_{1,5}^{(2)}(x) = \mathcal{O}(\mathrm{e}^{\beta\mu_4}) , \tag{4.99c}$$

$$b_{1,6}^{(2)}(x) = \mathcal{O}(\mathrm{e}^{\beta\mu_4}) , \qquad b_{1,1}^{(3)}(x) = \mathcal{O}(\mathrm{e}^{-\beta\mu_4}) , \qquad b_{1,2}^{(3)}(x) = \mathcal{O}(\mathrm{e}^{\beta\mu_4}) , \tag{4.99d}$$

$$b_{1,3}^{(3)}(x) = \mathcal{O}(\mathrm{e}^{\beta\mu_4}) , \qquad b_{1,4}^{(3)}(x) = \mathcal{O}(\mathrm{e}^{\beta\mu_4}) . \tag{4.99e}$$

Obviously, only seven of the auxiliary functions survive, and we can regard

$$b_{1,4}^{(1)}(x) \equiv b_{1,3}^{(2)}(x) \equiv b_{1,5}^{(2)}(x) \equiv b_{1,6}^{(2)}(x) \equiv b_{1,2}^{(3)}(x) \equiv b_{1,3}^{(3)}(x) \equiv b_{1,4}^{(3)}(x) \equiv 0 . \tag{4.100}$$

Since the function $b_{1,1}^{(3)}(x)$ diverges, we can also regard $b_{1,1}^{(3)}(x)/B_{1,1}^{(3)}(x) \to 1$. Therefore, the corresponding NLIE from the set (4.55) linearizes and can easily be solved analytically in Fourier space. This yields the result

$$\begin{aligned} \ln B_{1,1}^{(3)}(x) = -\beta\left(JD(x) - \frac{\mu_1+\mu_2+\mu_3}{3} + \mu_4\right) - \left[V_{[3]}^{(2)} * \ln\left(B_{1,1}^{(1)}B_{1,2}^{(1)}B_{1,3}^{(1)}\right)\right](x) \\ - \left[V_{[3]}^{(1)} * \ln\left(B_{1,1}^{(2)}B_{1,2}^{(2)}B_{1,4}^{(2)}\right)\right](x) , \end{aligned} \tag{4.101}$$

where

$$D(x) = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-|k|/2}}{\mathrm{e}^{-k}+1+\mathrm{e}^{k}} \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}k . \tag{4.102}$$

Substituting this result into the remaining NLIEs and relabelling $b_{1,4}^{(2)}(x)$ to $b_{1,3}^{(2)}(x)$, we arrive exactly at the coupled set of NLIEs of the $sl(3)$-symmetric case, see (4.23). As expected, in this sense the $sl(3)$-symmetric case is completely contained in the $sl(4)$ equations.

From the original set of auxiliary functions, we can even recover the explicit form of all $sl(3)$ auxiliary functions. Recall that the function $\lambda_4(x)$ vanishes in the limit $\mu_4 \to -\infty$. Therefore, we drop all Young tableaux containing a box with the number 4 from the auxiliary functions. The six remaining nonvanishing and nondivergent functions have exactly the form known from the definitions (4.17). Still, the definition of the λ functions is different for both models. Requiring, however, that the auxiliary functions are consistent with the previously gained set of coupled NLIEs, we conclude that $q_3(v) \to \phi_+(x)$ in the limit $\mu_4 \to \infty$.

Note that our choice of freezing out the state $\alpha = 4$ was completely arbitrary. Choosing one of the other states yields, after some relabelling of indices, the same NLIE and auxiliary functions. For $\mu_1 \to -\infty$ this implies $q_1(x) \to \phi_-(x)$, while for $\mu_\alpha \to \infty$ with $\alpha = 2$ or 3 we find $q_{\alpha-1}(x) \to q_\alpha(x)$.

The method is applicable also to the graded Uimin-Sutherland models. From the $sl(3|1)$-symmetric case, for example, it is possible to regain the NLIEs for either the $sl(3)$- or the $sl(2|1)$-symmetric case, depending on the basis state that is frozen out. Note that the $sl(2|2)$-symmetric case is a bit special, since here also auxiliary functions corresponding to a higher-level representation, where the first lower index is two, are used. Therefore, the reduction to the $sl(2|1)$-symmetric case, for example, gives the form of the NLIEs which will be introduced in Section 4.4 for the truncation level $m = 2$.

In order to generally determine which of the auxiliary functions survive if we take the limit $\mu_\alpha \to -\infty$ for some state α, it is in principle always necessary to consider the explicit asymptotics of the auxiliary functions. Based on the known results, however, we may conjecture the general structure of the selection rules. Let us stick to the $sl(n)$-symmetric case for a moment and consider the first set of auxiliary functions first. In all cases, where the auxiliary functions are known, exactly one of the auxiliary functions vanishes from this set for each choice of α. In fact, the first function vanishes for $\alpha = 1$, the second one for $\alpha = 2$ and so on. Now, let us assign a Young tableau with one box to each of the auxiliary functions, containing the corresponding number. If we move on to the second set of functions, we find that now every auxiliary function may vanish for two choices of α. The first one for $\alpha = 1$ and 2, the second one for $\alpha = 1$ and 3, etc. The pattern is exactly that of the admissible semi-standard $sl(n)$ Young tableaux with two boxes on top of each other. For $n \geq 4$, we would accordingly assign a three-box Young tableau to each auxiliary function of the third set. Indeed, the corresponding auxiliary functions vanish for three choices of α, and the patterns of the numbers coincide with the possible Young tableaux. It therefore seems reasonable to assign exactly one Young tableau to each auxiliary function. In order to determine if some auxiliary function survives when freezing out the state α, we just have to check if α is contained in the corresponding Young tableau. In the $sl(4)$-symmetric case, for example, we get the identifications,

$$b^{(1)}_{1,1}(x) \leftrightarrow \boxed{1}\,, \qquad b^{(1)}_{1,2}(x) \leftrightarrow \boxed{2}\,, \qquad b^{(1)}_{1,3}(x) \leftrightarrow \boxed{3}\,, \qquad b^{(1)}_{1,4}(x) \leftrightarrow \boxed{4}\,, \tag{4.103a}$$

$$b^{(2)}_{1,1}(x) \leftrightarrow \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\,, \qquad b^{(2)}_{1,2}(x) \leftrightarrow \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\,, \qquad b^{(2)}_{1,3}(x) \leftrightarrow \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}\,, \tag{4.103b}$$

$$b^{(2)}_{1,4}(x) \leftrightarrow \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\,, \qquad b^{(2)}_{1,5}(x) \leftrightarrow \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}\,, \qquad b^{(2)}_{1,6}(x) \leftrightarrow \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\,, \tag{4.103c}$$

$$b^{(3)}_{1,1}(x) \leftrightarrow \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\,, \qquad b^{(3)}_{1,2}(x) \leftrightarrow \begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array}\,, \qquad b^{(3)}_{1,3}(x) \leftrightarrow \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}\,, \qquad b^{(3)}_{1,4}(x) \leftrightarrow \begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}\,. \tag{4.103d}$$

In case of the $sl(n|1)$-symmetric cases, where we have a reduced number of auxiliary functions, we notice that the rule still holds, if we leave out the number of the graded state. The auxiliary functions of the $sl(2|1)$-symmetric case with grading $(+-+)$, for example, are thus connected to the tableaux

$$b^{(1)}_{1,1}(x) \leftrightarrow \boxed{1}\,, \qquad b^{(1)}_{1,2}(x) \leftrightarrow \boxed{3}\,, \qquad b^{(1)}_{2,1}(x) \leftrightarrow \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\,. \tag{4.104}$$

Unfortunately, although this conjecture is expected to hold for the general case and covers all the known results, the underlying representation theoretical argument is yet unknown.

In summary, we have seen that from a known set of NLIEs it is always possible to reconstruct the NLIEs for all models, which have a smaller number of basis states. Then the natural question arises whether it is also possible to go the opposite way and to introduce new basis states. For example, it would be nice to be able to directly derive the NLIEs of the $sl(5)$- from those of the $sl(4)$-symmetric case. Although the corresponding considerations do not lead to a complete set of NLIEs, it will be shown in Chapter 5 that they fix most of the structure of the NLIEs for higher-rank cases.

4.3.2 Zero-temperature limit and critical points

In order to treat the zero-temperature limit of the NLIEs, we define rescaled auxiliary functions by

$$e^{(a)}_{m,j}(x) = \frac{1}{\beta} \ln b^{(a)}_{m,j}(x) , \qquad E^{(a)}_{m,j}(x) = \frac{1}{\beta} \ln B^{(a)}_{m,j}(x) . \tag{4.105}$$

In the limit $T \to 0$ ($\beta \to \infty$) we get

$$E^{(a)}_{m,j}(x) \to e^{+(a)}_{m,j}(x) = \begin{cases} e^{(a)}_{m,j}(x) & \text{if } \Re(e^{(a)}_{m,j}(x)) > 0 \\ 0 & \text{if } \Re(e^{(a)}_{m,j}(x)) \leq 0 \end{cases} . \tag{4.106}$$

Therefore, auxiliary functions with negative real parts for all $x \in \mathbb{R}$ do no longer contribute, since $e^{+(a)}_{m,j}(x) \equiv 0$ for these functions.

As an example, we will again deal with the $sl(4)$-symmetric case of the Uimin-Sutherland model. Note that other cases can, in principle, be treated along the same lines. We assume $J > 0$ and, without loss of generality, choose the chemical potentials to be ordered, $\mu_1 \geq \mu_2 \geq \mu_3 \geq \mu_4$. Because all basis states are equivalent, changing this order just amounts to some permutation of indices in the following calculations. We observe that only one auxiliary function remains from each representation. In our case, the functions $e^{(a)}_{1,1}(x)$ for $a = 1, 2, 3$ survive. The corresponding NLIEs linearize and take the form

$$e^{(a)}_{1,1}(x) = -JV^{(a)}_{[4]}(x) - c^{(a)}_1 - \sum_{b=1}^{3} \left[\mathcal{K}^{(a,b)}_{[4]} * e^{+(b)}_{1,1} \right] (x) . \tag{4.107}$$

It follows that the remaining auxiliary functions are real and even. The ground-state energy is given by

$$f_0 = J \left(1 - \frac{\pi}{4} - \frac{3}{2} \ln 2 \right) - \frac{1}{4} \sum_{j=1} \mu_j - \sum_{a=1}^{3} \left[V^{(a)}_{[4]} * e^{+(a)}_{1,1} \right] (0) . \tag{4.108}$$

We like to stress that these are exactly the equations, which also follow from the zero-temperature limit of the TBA equations, confer (3.43) for $n = 4$. They may alternatively be obtained directly from the Bethe ansatz equations for the Hamiltonian (2.4), see [34].

If all chemical potentials are equal, these equations have a particularly simple solution. In this case, we get $e^{(a)}_{1,1}(x) = -JV^{(a)}_{[4]}(x)$ and $e^{+(a)}_{1,1}(x) \equiv 0$ so that the ground state energy is just $f_0 = J(1 - \frac{\pi}{4} - \frac{3}{2} \ln 2) - \mu_1$. In general, depending on certain differences of the chemical potentials, the ground state can be in one of four possible phases, admitting certain simplifications to the NLIEs (4.107). We start with the phase, where all degrees of freedom are frozen out, i.e. only the state $\alpha = 1$ survives. In this case, we have $e^{(a)}_{1,1}(x) = e^{+(a)}_{1,1}(x)$ for all a. As a consequence, the equations (4.107) can be solved analytically. We find the restriction $\mu_1 - \mu_2 \geq 4J$ and obtain

$$e^{(1)}_{1,1}(x) = \mu_1 - \mu_2 - J \frac{4}{4x^2 + 1} , \qquad e^{(2)}_{1,1}(x) = \mu_2 - \mu_3 , \qquad e^{(3)}_{1,1}(x) = \mu_3 - \mu_4 , \tag{4.109}$$

while the ground-state energy turns out to be $f_0 = J - \mu_1$. As expected, the ground state is fully polarized.

Below the first critical point, that is for $\mu_1 - \mu_2 < 4J$, the function $e^{(1)}_{1,1}(x)$ possesses two real roots, and therefore $e^{(a)}_{1,1}(x) = e^{+(a)}_{1,1}(x)$ remains valid only for $a = 2, 3$. We can still solve the equations (4.107) for the latter two functions to obtain

$$e^{(1)}_{1,1}(x) = -JV^{(1)}_{[2]}(x) + \frac{\mu_1 - \mu_2}{2} - \left[\mathcal{K}^{(1,1)}_{[2]} * e^{+(1)}_{1,1}\right](x)\,, \tag{4.110a}$$

$$e^{(2)}_{1,1}(x) = J\mathcal{K}^{(1,1)}_{[2]}(x) + \frac{1}{2}\sum_{j=1}^{2}(\mu_j - \mu_3) - \left[V^{(1)}_{[2]} * e^{+(1)}_{1,1}\right](x)\,, \tag{4.110b}$$

$$e^{(3)}_{1,1}(x) = \mu_3 - \mu_4\,. \tag{4.110c}$$

For the ground-state energy, we arrive at

$$f_0 = J(1 - 2\ln 2) - \frac{\mu_1 + \mu_2}{2} - \left[V^{(1)}_{[2]} * e^{+(1)}_{1,1}\right](0)\,. \tag{4.111}$$

This is exactly the $T = 0$ behaviour of the $sl(2)$-symmetric case. Two states, $\alpha = 1$ and 2, are present in the ground state. Note that these equations are only valid above the second critical point, i.e. as long as $e^{(2)}_{1,1}(x) \geq 0$ for all $x \in \mathbb{R}$. From equation (4.110b), we find the restriction

$$\sum_{j=1}^{2}(\mu_j - \mu_3) \geq 4J\ln 2 + 2\left[V^{(1)}_{[2]} * e^{+(1)}_{1,1}\right](0)\,, \tag{4.112}$$

where the positive convolution term unfortunately still depends on the function $e^{+(1)}_{1,1}(x)$, which is not explicitly known. The convolution term vanishes for $\mu_1 = \mu_2$.

Below the second critical point, the state $\alpha = 3$ also contributes to the ground-state. Both functions $e^{(1)}_{1,1}$ and $e^{(2)}_{1,1}$ possess two real roots, and only $e^{(3)}_{1,1}(x) = e^{+(3)}_{1,1}(x)$ remains valid. Here, we recover the $T = 0$ behaviour of the $sl(3)$-symmetric case of the Uimin-Sutherland model,

$$e^{(a)}_{1}(x) = -JV^{(a)}_{[3]}(x) - c^{(a)}_{1} - \sum_{b=1}^{2}\big[K^{(a,b)}_{[3]} * e^{+(b)}_{1}\big](x) \qquad (a = 1, 2)\,, \tag{4.113a}$$

$$e^{(3)}_{1}(x) = -JW(x) + \frac{1}{3}\sum_{j=1}^{3}(\mu_j - \mu_4) - \sum_{b=1}^{2}\big[V^{(3-b)}_{[3]} * e^{+(b)}_{1}\big](x)\,, \tag{4.113b}$$

where $c^{(a)}_{1}$ are the constants of the $sl(3)$-symmetric case defined in equation (4.25). The ground-state energy can be calculated by use of

$$f_0 = J\left(1 - \frac{\pi}{3\sqrt{3}} - \ln 3\right) - \frac{1}{3}\sum_{j=1}^{3}\mu_j - \sum_{a=1}^{2}\big[V^{(a)}_{[3]} * e^{+(a)}_{1,1}\big](0)\,. \tag{4.114}$$

These equations hold as long as we are above the third and last critical point. From equation (4.113b) we get the restriction

$$\sum_{j=1}^{3}(\mu_j - \mu_4) \geq J(\pi\sqrt{3} - 3\ln 3) + 3\sum_{a=1}^{2}\big[V_{[3]}^{(3-a)} * e^{+}{}_{1,1}^{(a)}\big](0)\,, \tag{4.115}$$

where we again have no explicit expression for the positive convolution terms, which vanish if $\mu_1 = \mu_2 = \mu_3$.

Only below the third critical point, all four basis states contribute to the ground state. Here, equation (4.107) can not be further simplified.

Note that the other cases of the Uimin-Sutherland model may be treated in basically the same way. However, like in the previous section, it is important to know, which of the auxiliary functions still contribute in the zero-temperature limit depending on some given ordering of the general chemical potentials. In order to find this out, the explicit asymptotics of the functions have to be examined. However, based on the Young tableaux that we have assigned to each auxiliary function in Section 4.3.1, we can again conjecture the selection rules. Let us deal with the $sl(n)$-symmetric case first. Suppose, we have the order $\mu_{\alpha_1} \geq \mu_{\alpha_2} \geq \ldots \geq \mu_{\alpha_n}$. Then, in the limit $T \to 0$, we expect only those auxiliary functions to contribute to the largest eigenvalue, whose corresponding Young tableaux from top to bottom contain exactly the first numbers of the sequence $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Accordingly, only one auxiliary function from each set will survive. This makes sense, since the remaining auxiliary functions and linearized integral equations should be equivalent to those obtained by the TBA approach. The number of remaining functions therefore has to be equal to the rank of the algebra $sl(n)$. Similar arguments are expected to hold for the $sl(n|1)$-symmetric case. Here, the single auxiliary function from the last set will always contribute, while from the remaining sets only one function is selected, according to the arguments above, but where the general chemical potential of the state with grading $p(\alpha_j) = 1$ is left out.

4.4 Connection to the TBA approach

For each of the cases treated in the previous sections, we have pointed out the connection of the auxiliary functions we used to those of the TBA approach. Let us further explore the relationship between both approaches. It has first been pointed out by Suzuki [76] that, after some slight modification, the auxiliary functions of the $sl(2)$-symmetric case can be used to exactly truncate the TBA equations at some arbitrary fusion level, leaving only a finite number of NLIEs. Note that in this sense, the auxiliary functions (4.1) make for the natural truncation at level $m = 1$. In the following sections, we will show how to extend this approach to higher-rank cases of the Uimin-Sutherland model.

4.4.1 Exact truncation of the TBA equations for the $sl(2)$ case

Recall that the TBA approach for the $sl(2)$-symmetric case of the Uimin-Sutherland model deals with infinitely many auxiliary functions $y_k^{(1)}(x)$, where $k = 1, \ldots, \infty$. Let us truncate the set of functions at some arbitrary level $k = m$. This can be done by considering only $m-1$ many of the original auxiliary functions and replacing the mth function by the two new functions

$$b^{(1)}_{m,1}(x) = \frac{\lambda_1(x+\frac{m}{2}\mathrm{i})\Lambda^{(1)}_{m-1}(x)}{\prod_{k=0}^{m-1}\lambda_2(x-\frac{2k-m}{2}\mathrm{i})}\,, \qquad b^{(1)}_{m,2}(x) = \frac{\Lambda^{(1)}_{m-1}(x)\lambda_2(x-\frac{m}{2}\mathrm{i})}{\prod_{k=1}^{m}\lambda_1(x-\frac{2k-m}{2}\mathrm{i})}\,. \tag{4.116}$$

Then the new uppercase functions $B^{(1)}_{m,j}(x) = b^{(1)}_{m,j}(x) + 1$ can be written as

$$B^{(1)}_{m,1}(x) = \frac{\Lambda^{(1)}_{m}(x+\frac{\mathrm{i}}{2})}{\prod_{k=0}^{m-1}\lambda_2(x-\frac{2k-m}{2}\mathrm{i})}\,, \qquad B^{(1)}_{m,2}(x) = \frac{\Lambda^{(1)}_{m}(x-\frac{\mathrm{i}}{2})}{\prod_{k=1}^{m}\lambda_1(x-\frac{2k-m}{2}\mathrm{i})}\,. \tag{4.117}$$

This definition leads to the following factorized form of the auxiliary functions:

$$b^{(1)}_{m,1}(x) = \frac{q_1(x+\frac{m+2}{2}\mathrm{i})\widetilde{\Lambda}^{(1)}_{m-1}(x)}{\phi_-(x+\frac{m}{2}\mathrm{i})\phi_+(x+\frac{m+2}{2}\mathrm{i})q_1(x-\frac{m}{2}\mathrm{i})} \cdot \frac{\mathrm{e}^{\beta\mu_1}a^{(1)}_{m-1}}{\mathrm{e}^{m\beta\mu_2}}\,, \tag{4.118a}$$

$$b^{(1)}_{m,2}(x) = \frac{q_1(x-\frac{m+2}{2}\mathrm{i})\widetilde{\Lambda}^{(1)}_{m-1}(x)}{\phi_+(x-\frac{m}{2}\mathrm{i})\phi_-(x-\frac{m+2}{2}\mathrm{i})q_1(x+\frac{m}{2}\mathrm{i})} \cdot \frac{\mathrm{e}^{\beta\mu_2}a^{(1)}_{m-1}}{\mathrm{e}^{m\beta\mu_1}}\,, \tag{4.118b}$$

$$B^{(1)}_{m,1}(x) = \frac{q_1(x+\frac{m}{2}\mathrm{i})\widetilde{\Lambda}^{(1)}_{m}(x+\frac{\mathrm{i}}{2})}{\phi_-(x+\frac{m}{2}\mathrm{i})\phi_+(x+\frac{m+2}{2}\mathrm{i})q_1(x-\frac{m}{2}\mathrm{i})} \cdot \frac{a^{(1)}_{m}}{\mathrm{e}^{m\beta\mu_2}}\,, \tag{4.118c}$$

$$B^{(1)}_{m,2}(x) = \frac{q_1(x-\frac{m}{2}\mathrm{i})\widetilde{\Lambda}^{(1)}_{m}(x-\frac{\mathrm{i}}{2})}{\phi_+(x-\frac{m}{2}\mathrm{i})\phi_-(x-\frac{m+2}{2}\mathrm{i})q_1(x+\frac{m}{2}\mathrm{i})} \cdot \frac{a^{(1)}_{m}}{\mathrm{e}^{m\beta\mu_1}}\,. \tag{4.118d}$$

From the latter form of the functions it is immediately clear, that the function $Y^{(1)}_m(x)$ can be written as a product of the new uppercase functions,

$$Y^{(1)}_m(x) = B^{(1)}_{m,1}(x)B^{(1)}_{m,2}(x)\,, \tag{4.119}$$

which is the generalization of relation (4.16). However, we also find an additional relation similar to (3.16),

$$b^{(1)}_{m,1}(x-\mathrm{i}/2)b^{(1)}_{m,2}(x+\mathrm{i}/2) = Y^{(1)}_{m-1}(x)\,. \tag{4.120}$$

The advantage compared to the usual set of TBA functions is the fact that the set of auxiliary functions $y^{(1)}_k(x)$, $b^{(1)}_{m,1}(x)$, $b^{(1)}_{m,2}(x)$ no longer contains the eigenvalue function $\widetilde{\Lambda}^{(1)}_{m+1}(x)$ by avoiding the explicit use of $y^{(1)}_m(x)$. Instead, we now have $m+1$ many auxiliary functions exactly meeting the number of unknown functions which are given by the eigenvalues $\widetilde{\Lambda}^{(1)}_k(x)$ for $k = 1, \ldots, m$ and the function $q_1(x)$. Thus, the former infinite set of functions has been exactly closed.

Note that the structure of the factorized form of the auxiliary functions given by equation (4.118) is very similar to that of the original auxiliary functions (4.2). In each fusion step, the singularities of the auxiliary functions are shifted away from the real axis by an amount of i/2. Thereby, the eigenvalue functions $\phi_-(x-\mathrm{i}/2)\phi_+(x+\mathrm{i}/2) = \widetilde{\Lambda}_0^{(1)}(x)$ and $\widetilde{\Lambda}_1^{(1)}(x)$ are replaced by $\widetilde{\Lambda}_{m-1}^{(1)}(x)$ and $\widetilde{\Lambda}_m^{(1)}(x)$, respectively, while the other functions $\phi_-(x)$, $\phi_+(x)$ and $q_1(x)$ remain structurally unchanged, so that there is just a shift in the argument.

Now, the final coupled set of NLIEs can be calculated from the auxiliary functions by the usual method. For the first, $m-1$ many auxiliary functions $y_j^{(1)}(x)$, we get

$$\ln y_k^{(1)}(x) = -J\beta\delta_{k,1}V_{[2]}^{(1)}(x) + \left[V_{[2]}^{(1)} * \Big((1-\delta_{k,1})\ln Y_{k-1}^{(1)} + \ln Y_{k+1}^{(1)}\Big)\right](x)\,, \tag{4.121}$$

where we use equation (4.119) in order to determine the function $Y_m^{(1)}(x)$ that appears for $k = m-1$. The remaining NLIEs for the functions $b_{m,j}^{(1)}(x)$ are

$$\ln b_{m,j}^{(1)}(x) = -\beta c_j^{(1)} + \left[V_{[2]}^{(1)} * \ln Y_{m-1}^{(1)}\right](x) - \sum_{k=1}^{2}\left[\mathbf{K}_{j,k}^{(1,1)} * \ln B_{m,k}^{(1)}\right](x)\,, \tag{4.122}$$

where the kernel matrix is given by

$$\mathbf{K}^{(1,1)}(x) = \begin{pmatrix} K_0(x) & K_1(x) \\ K_2(x) & K_0(x) \end{pmatrix}\,, \tag{4.123}$$

and the kernels $K_j(x)$ and constants $c_j^{(1)}$ are defined like in Section 4.1.1, see equations (4.10) and (4.12). The largest eigenvalue of the QTM follows from

$$\ln \Lambda_1^{(1)}(0) = -\beta\left\{J(1-2\ln 2) - \frac{\mu_1+\mu_2}{2}\right\} + \left[V_{[2]}^{(1)} * \ln Y_1^{(1)}\right](0)\,. \tag{4.124}$$

4.4.2 Exact truncation of the TBA equations for the *sl*(3) case

In the TBA approach for the $sl(3)$-symmetric case of Uimin-Sutherland model, there are infinitely many auxiliary functions of two types, $y_k^{(1)}(x)$ and $y_k^{(2)}(x)$, with $k = 1,\ldots,\infty$. In analogy to the $sl(2)$ case, it should be possible to exactly truncate both sets of functions at some arbitrary step using six auxiliary functions similar to (4.17) to replace $y_m^{(1)}(x)$ and $y_m^{(2)}(x)$. Indeed, this is possible. We find

$$b_{m,1}^{(1)}(x) = \frac{\lambda_1^{(1)}(x+\frac{m}{2}\mathrm{i})\Lambda_{m-1}^{(1)}(x)}{\Gamma_{m,2}^{(1)}(x+\frac{\mathrm{i}}{2})}\,, \tag{4.125a}$$

$$b_{m,2}^{(1)}(x) = \frac{\lambda_1^{(2)}(x+\frac{m-1}{2}\mathrm{i})\Lambda_{m-1}^{(1)}(x)\Lambda_{m-1}^{(3)}(x)\lambda_3^{(2)}(x-\frac{m-1}{2}\mathrm{i})}{\prod_{k=1}^{m}\lambda_2^{(2)}(x-\frac{2k-m-1}{2}\mathrm{i})\Lambda_m^{(2)}(x)}\,, \tag{4.125b}$$

$$b_{m,3}^{(1)}(x) = \frac{\Lambda_{m-1}^{(1)}(x)\lambda_3^{(1)}(x-\frac{m}{2}\mathrm{i})}{\Gamma_{m,1}^{(1)}(x-\frac{\mathrm{i}}{2})}\,, \tag{4.125c}$$

$$b_{m,1}^{(2)}(x) = \frac{\lambda_1^{(2)}(x+\frac{m}{2}\mathrm{i})\Lambda_{m-1}^{(2)}(x)}{\Gamma_{m,2}^{(2)}(x+\frac{\mathrm{i}}{2})}\,, \tag{4.125d}$$

$$b^{(2)}_{m,2}(x) = \frac{\lambda^{(1)}_1(x+\frac{m-1}{2}\mathrm{i})\Lambda^{(2)}_{m-1}(x)\lambda^{(1)}_3(x-\frac{m-1}{2}\mathrm{i})}{\prod_{k=1}^m \lambda^{(1)}_2(x-\frac{2k-m-1}{2}\mathrm{i})\Lambda^{(1)}_m(x)}\,, \tag{4.125e}$$

$$b^{(2)}_{m,3}(x) = \frac{\Lambda^{(2)}_{m-1}(x)\lambda^{(2)}_3(x-\frac{m}{2}\mathrm{i})}{\Gamma^{(2)}_{m,1}(x-\frac{\mathrm{i}}{2})}\,, \tag{4.125f}$$

where we have defined

$$\lambda^{(1)}_j(x) = \lambda_j(x)\,, \qquad \lambda^{(2)}_j(x) = \begin{cases}\lambda_1(x-\mathrm{i}/2)\lambda_j(x+\mathrm{i}/2) & \text{for } j=1,2\\ \lambda_2(x-\mathrm{i}/2)\lambda_3(x+\mathrm{i}/2) & \text{for } j=3\end{cases} \tag{4.126}$$

and

$$\Gamma^{(a)}_{m,1}(x) = \sum_{k=0}^m \left\{\prod_{j=1}^k \lambda^{(a)}_1\left(x-\frac{2j-m-1}{2}\mathrm{i}\right)\prod_{j=k+1}^m \lambda^{(a)}_2\left(x-\frac{2j-m-1}{2}\mathrm{i}\right)\right\}\,, \tag{4.127a}$$

$$\Gamma^{(a)}_{m,2}(x) = \sum_{k=0}^m \left\{\prod_{j=1}^k \lambda^{(a)}_2\left(x-\frac{2j-m-1}{2}\mathrm{i}\right)\prod_{j=k+1}^m \lambda^{(a)}_3\left(x-\frac{2j-m-1}{2}\mathrm{i}\right)\right\}\,. \tag{4.127b}$$

For the uppercase auxiliary functions $B^{(a)}_{m,j}(x) = b^{(a)}_{m,j}(x)+1$, we find the expressions

$$B^{(1)}_{m,1}(x) = \frac{\Lambda^{(1)}_m(x+\frac{\mathrm{i}}{2})}{\Gamma^{(1)}_{m,2}(x+\frac{\mathrm{i}}{2})}\,, \tag{4.128a}$$

$$B^{(1)}_{m,2}(x) = \frac{\Gamma^{(2)}_{m,1}(x)\Gamma^{(2)}_{m,2}(x)}{\prod_{k=1}^m \lambda^{(2)}_2(x-\frac{2k-m-1}{2}\mathrm{i})\Lambda^{(2)}_m(x)}\,, \tag{4.128b}$$

$$B^{(1)}_{m,3}(x) = \frac{\Lambda^{(1)}_m(x-\frac{\mathrm{i}}{2})}{\Gamma^{(1)}_{m,1}(x-\frac{\mathrm{i}}{2})}\,, \tag{4.128c}$$

$$B^{(2)}_{m,1}(x) = \frac{\Lambda^{(2)}_m(x+\frac{\mathrm{i}}{2})}{\Gamma^{(2)}_{m,2}(x+\frac{\mathrm{i}}{2})}\,, \tag{4.128d}$$

$$B^{(2)}_{m,2}(x) = \frac{\Gamma^{(1)}_{m,1}(x)\Gamma^{(1)}_{m,2}(x)}{\prod_{k=1}^m \lambda^{(1)}_2(x-\frac{2k-m-1}{2}\mathrm{i})\Lambda^{(1)}_m(x)}\,, \tag{4.128e}$$

$$B^{(2)}_{m,3}(x) = \frac{\Lambda^{(2)}_m(x-\frac{\mathrm{i}}{2})}{\Gamma^{(2)}_{m,1}(x-\frac{\mathrm{i}}{2})}\,. \tag{4.128f}$$

The results for $B^{(a)}_{m,1}(x)$ and $B^{(a)}_{m,3}(x)$ are a direct consequence of the relation

$$\begin{aligned}\Lambda^{(a)}_m(x) &= \Gamma^{(a)}_{m,1}(x) + \Lambda^{(a)}_{m-1}\left(x+\frac{\mathrm{i}}{2}\right)\lambda^{(a)}_3\left(x-\frac{m-1}{2}\mathrm{i}\right)\\ &= \lambda^{(a)}_1\left(x+\frac{m-1}{2}\mathrm{i}\right)\Lambda^{(a)}_{m-1}\left(x-\frac{\mathrm{i}}{2}\right)+\Gamma^{(a)}_{m,2}(x)\,,\end{aligned} \tag{4.129}$$

which follows from the structure of the fused eigenvalue functions (3.3), while the results for the remaining functions $B^{(a)}_{m,2}(x)$ can be proved by induction.

For the functions $\Gamma^{(a)}_{m,j}(x)$, we find the factorized form

$$\Gamma^{(1)}_{m,1}(x) = n^{(1)}_m(x)\cdot\frac{\phi_+(x-\frac{m-1}{2}\mathrm{i})X_{m,1}(x)}{q_2(x-\frac{m-1}{2}\mathrm{i})}\cdot\chi_{m,1}\,, \tag{4.130a}$$

$$\Gamma^{(1)}_{m,2}(x) = n^{(1)}_m(x)\cdot\frac{\phi_-(x+\frac{m-1}{2}\mathrm{i})X_{m,2}(x)}{q_1(x+\frac{m-1}{2}\mathrm{i})}\cdot\chi_{m,2}\,, \tag{4.130b}$$

$$\Gamma^{(2)}_{m,1}(x) = n^{(2)}_m(x)\cdot\frac{\phi_-(x-\frac{m+2}{2}\mathrm{i})X_{m,2}(x+\frac{\mathrm{i}}{2})}{q_1(x-\frac{m}{2}\mathrm{i})}\cdot\mathrm{e}^{m\beta\mu_1}\chi_{m,2}\,, \tag{4.130c}$$

$$\Gamma^{(2)}_{m,2}(x) = n^{(2)}_m(x)\cdot\frac{\phi_+(x+\frac{m+2}{2}\mathrm{i})X_{m,1}(x-\frac{\mathrm{i}}{2})}{q_2(x+\frac{m}{2}\mathrm{i})}\cdot\mathrm{e}^{m\beta\mu_3}\chi_{m,1}\,, \tag{4.130d}$$

where the functions $X_{m,j}(x)$ are just polynomials of degree N. This fact may also be proved by induction, where one exploits that the special cases $X_{1,j}(x)$ are identical to the polynomials of hole solutions $q^{(h)}_j(x)$. Moreover, we have checked numerically that the roots of the functions $X_{m,j}(x)$ have imaginary parts close to $\pm(m+1)/2$. The constants are explicitly given by

$$\chi_{m,1} = \sum_{k=0}^{m}\mathrm{e}^{k\beta\mu_1}\mathrm{e}^{(m-k)\beta\mu_2}\,, \qquad \chi_{m,2} = \sum_{k=0}^{m}\mathrm{e}^{k\beta\mu_2}\mathrm{e}^{(m-k)\beta\mu_3}\,. \tag{4.131}$$

Using this information we are able factorize the auxiliary functions. We get the result

$$b^{(1)}_{m,1}(x) = \frac{q_1(x+\frac{m+2}{2}\mathrm{i})\widetilde{\Lambda}^{(1)}_{m-1}(x)}{\phi_-(x+\frac{m}{2}\mathrm{i})X_{m,2}(x+\frac{\mathrm{i}}{2})}\cdot\frac{\mathrm{e}^{\beta\mu_1}a^{(1)}_{m-1}}{\chi_{m,2}}\,, \tag{4.132a}$$

$$b^{(1)}_{m,2}(x) = \frac{q_1(x-\frac{m+2}{2}\mathrm{i})q_2(x+\frac{m+2}{2}\mathrm{i})\widetilde{\Lambda}^{(1)}_{m-1}(x)}{q_1(x+\frac{m}{2}\mathrm{i})q_2(x-\frac{m}{2}\mathrm{i})\widetilde{\Lambda}^{(2)}_{m}(x)}\cdot\frac{\mathrm{e}^{(m+1)\beta\mu_2}a^{(1)}_{m-1}}{a^{(2)}_m}\,, \tag{4.132b}$$

$$b^{(1)}_{m,3}(x) = \frac{q_2(x-\frac{m+2}{2}\mathrm{i})\widetilde{\Lambda}^{(1)}_{m-1}(x)}{\phi_+(x-\frac{m}{2}\mathrm{i})X_{m,1}(x-\frac{\mathrm{i}}{2})}\cdot\frac{\mathrm{e}^{\beta\mu_3}a^{(1)}_{m-1}}{\chi_{m,1}}\,, \tag{4.132c}$$

$$b^{(2)}_{m,1}(x) = \frac{q_2(x+\frac{m+3}{2}\mathrm{i})\widetilde{\Lambda}^{(2)}_{m-1}(x)}{\phi_+(x+\frac{m+3}{2}\mathrm{i})X_{m,1}(x)}\cdot\frac{\mathrm{e}^{\beta(\mu_1+\mu_2)}a^{(2)}_{m-1}}{\mathrm{e}^{m\beta\mu_3}\chi_{m,1}}\,, \tag{4.132d}$$

$$b^{(2)}_{m,2}(x) = \frac{q_1(x+\frac{m+1}{2}\mathrm{i})q_2(x-\frac{m+1}{2}\mathrm{i})\widetilde{\Lambda}^{(2)}_{m-1}(x)}{q_1(x-\frac{m+1}{2}\mathrm{i})q_2(x+\frac{m+1}{2}\mathrm{i})\widetilde{\Lambda}^{(1)}_{m}(x)}\cdot\frac{\mathrm{e}^{\beta(\mu_1+\mu_3)}a^{(2)}_{m-1}}{\mathrm{e}^{m\beta\mu_2}a^{(1)}_m}\,, \tag{4.132e}$$

$$b^{(2)}_{m,3}(x) = \frac{q_1(x-\frac{m+3}{2}\mathrm{i})\widetilde{\Lambda}^{(2)}_{m-1}(x)}{\phi_-(x-\frac{m+3}{2}\mathrm{i})X_{m,2}(x)}\cdot\frac{\mathrm{e}^{\beta(\mu_2+\mu_3)}a^{(2)}_{m-1}}{\mathrm{e}^{m\beta\mu_1}\chi_{m,2}}\,. \tag{4.132f}$$

The uppercase auxiliary functions may be written as

$$B^{(1)}_{m,1}(x) = \frac{q_1(x+\frac{m}{2}\mathrm{i})\widetilde{\Lambda}^{(1)}_{m}(x+\frac{\mathrm{i}}{2})}{\phi_-(x+\frac{m}{2}\mathrm{i})X_{m,2}(x+\frac{\mathrm{i}}{2})}\cdot\frac{a^{(1)}_m}{\chi_{m,2}}\,, \tag{4.133a}$$

$$B^{(1)}_{m,2}(x) = \frac{X_{m,1}(x-\frac{\mathrm{i}}{2})X_{m,2}(x+\frac{\mathrm{i}}{2})}{q_1(x+\frac{m}{2}\mathrm{i})q_2(x-\frac{m}{2}\mathrm{i})\widetilde{\Lambda}^{(2)}_m(x)} \cdot \frac{\chi_{m,1}\chi_{m,2}}{a^{(2)}_m}\,, \tag{4.133b}$$

$$B^{(1)}_{m,3}(x) = \frac{q_2(x-\frac{m}{2}\mathrm{i})\widetilde{\Lambda}^{(1)}_m(x-\frac{\mathrm{i}}{2})}{\phi_+(x-\frac{m}{2}\mathrm{i})X_{m,1}(x-\frac{\mathrm{i}}{2})} \cdot \frac{a^{(1)}_m}{\chi_{m,1}}\,, \tag{4.133c}$$

$$B^{(2)}_{m,1}(x) = \frac{q_2(x+\frac{m+1}{2}\mathrm{i})\widetilde{\Lambda}^{(2)}_m(x+\frac{\mathrm{i}}{2})}{\phi_+(x+\frac{m+3}{2}\mathrm{i})X_{m,1}(x)} \cdot \frac{a^{(2)}_m}{\mathrm{e}^{m\beta\mu_3}\chi_{m,1}}\,, \tag{4.133d}$$

$$B^{(2)}_{m,2}(x) = \frac{X_{m,1}(x)X_{m,2}(x)}{q_1(x-\frac{m+1}{2}\mathrm{i})q_2(x+\frac{m+1}{2}\mathrm{i})\widetilde{\Lambda}^{(1)}_m(x)} \cdot \frac{\chi_{m,1}\chi_{m,2}}{\mathrm{e}^{m\beta\mu_2}a^{(1)}_m}\,, \tag{4.133e}$$

$$B^{(2)}_{m,3}(x) = \frac{q_1(x-\frac{m+1}{2}\mathrm{i})\widetilde{\Lambda}^{(2)}_m(x-\frac{\mathrm{i}}{2})}{\phi_-(x-\frac{m+3}{2}\mathrm{i})X_{m,2}(x)} \cdot \frac{a^{(2)}_m}{\mathrm{e}^{m\beta\mu_1}\chi_{m,2}}\,. \tag{4.133f}$$

From the latter equations, we immediately find the connection to the auxiliary functions $Y^{(a)}_m(x)$ in analogy to equation (4.18),

$$Y^{(a)}_m(x) = B^{(a)}_{m,1}(x)B^{(a)}_{m,2}(x)B^{(a)}_{m,3}(x)\,, \tag{4.134}$$

for $a = 1$ and 2. Like in the $sl(2)$-symmetric case, we also find further relations which are similar to equation (3.16),

$$b^{(1)}_{m,1}(x-\mathrm{i}/2)b^{(1)}_{m,3}(x+\mathrm{i}/2) = \frac{Y^{(1)}_{m-1}(x)}{\overline{B}^{(2)}_{m,2}(x)}\,, \tag{4.135a}$$

$$b^{(2)}_{m,1}(x-\mathrm{i}/2)b^{(2)}_{m,3}(x+\mathrm{i}/2) = \frac{Y^{(2)}_{m-1}(x)}{\overline{B}^{(1)}_{m,2}(x)}\,, \tag{4.135b}$$

where $\overline{B}^{(a)}_{m,2}(x) = B^{(a)}_{m,2}(x)/b^{(a)}_{m,2}(x)$.

Like the original TBA auxiliary functions, all additional auxiliary functions have the desired ANZC property, and thus the coupled set of NLIEs can be derived via the standard approach. In comparison to the original auxiliary functions for $m = 1$, confer equations (4.73) and (4.74), the eigenvalue functions $\phi_-(x-\frac{a+1}{2}\mathrm{i})\phi_+(x+\frac{a+1}{2}\mathrm{i}) = \widetilde{\Lambda}^{(a)}_0(x)$ and $\widetilde{\Lambda}^{(a)}_1(x)$ are replaced by $\widetilde{\Lambda}^{(a)}_{m-1}(x)$ and $\widetilde{\Lambda}^{(a)}_m(x)$, respectively, the hole-solution polynomials $q^{(h)}_j(x)$ are replaced by $X_{m,j}(x)$, and all other roots and poles of the auxiliary functions stemming from $\phi_-(x)$, $\phi_+(x)$ and $q_j(x)$ are just shifted away from the real axis by $\mathrm{i}/2$ in each fusion step. Like in the $sl(2)$-symmetric case, these are only small, rather systematic modifications. In the Young tableaux formulation, however, this would not have been obvious.

Applying the standard derivation utilizing the logarithmic Fourier transform of the derivatives, we find the final coupled set of NLIEs to be

$$\ln y^{(a)}_k(x) = -J\beta\delta_{k,1}V^{(a)}_{[3]}(x) + \sum_{b=1}^{2}\Big\{\Big[A^{(a,b)}_{[n]} * \Big((1-\delta_{k,1})\ln Y^{(b)}_{k-1} + \ln Y^{(b)}_{k+1}\Big)\Big](x)$$

$$- \left[\left(A_{[n]}^{(a,b-1)} + A_{[n]}^{(a,b+1)} \right) * \ln Y_k^{(b)} \right] (x) \Bigg\} , \tag{4.136a}$$

$$\ln b_{m,j}^{(a)}(x) = -\beta c_j^{(a)} + \sum_{b=1}^{2} \left\{ \left[A_{[n]}^{(a,b)} * \ln Y_{m-1}^{(b)} \right] (x) - \sum_{k=1}^{3} \left[\mathbf{K}_{j,k}^{(a,b)} * \ln B_{m,k}^{(b)} \right] (x) \right\} , \tag{4.136b}$$

where the function $A_{[n]}^{(a,b)}(x)$ is the one introduced in Section 3.4.2, confer equation (3.26), and the kernel matrices $\mathbf{K}^{(a,b)}(x)$ and the constants $c_j^{(a)}$ are those introduced in Section 4.1.2, confer equations (4.26) and (4.25). The function $Y_m^{(a)}(x)$, which appears in the first set of equations for $k = m-1$, is obtained via equation (4.134). Note that the first set of equations is equivalent to first $m-1$ many TBA equations (3.25) for $n = 3$, while the second set is very similar to the previously derived NLIEs (4.23). Finally, the equation for the largest eigenvalue of the QTM is

$$\ln \Lambda_1^{(1)}(0) = -\beta \left\{ J \left(1 - \frac{\pi}{3\sqrt{3}} - \ln 3 \right) - \frac{1}{3} \sum_{j=1}^{3} \mu_j \right\} + \sum_{a=1}^{2} \left[V^{(a)} * \ln Y_1^{(a)} \right] (0) . \tag{4.137}$$

4.4.3 Exact truncation of the TBA equations for the $sl(2|1)$ case

Let us now turn to the $sl(2|1)$-symmetric case of the Uimin-Sutherland model. One family of TBA auxiliary functions, $y_j^{(1)}(x)$, has infinitely many members, $j = 1, \ldots, \infty$, while the function $y_1^{(2)}(x)$ is unique. Therefore, the structure is very similar to the $sl(2)$-symmetric case. We choose the grading $(+ - +)$ and truncate the equations at $j = m$ by replacing $y_m^{(1)}(x)$ by the two new auxiliary functions

$$b_{m,1}^{(1)}(x) = \frac{\lambda_1(x + \frac{m}{2}\mathrm{i})\Lambda_{m-1}^{(1)}(x)}{\Gamma_{m,2}(x + \frac{\mathrm{i}}{2})} , \qquad b_{m,2}^{(1)}(x) = \frac{\Lambda_{m-1}^{(1)}(x)\lambda_3(x - \frac{m}{2}\mathrm{i})}{\Gamma_{m,1}(x - \frac{\mathrm{i}}{2})} , \tag{4.138}$$

where we have defined

$$\Gamma_{m,1}(x) = \prod_{k=1}^{m-1} \lambda_1 \left(x - \frac{2k-m-1}{2}\mathrm{i} \right) \left[\lambda_1 \left(x - \frac{m-1}{2}\mathrm{i} \right) + \lambda_2 \left(x - \frac{m-1}{2} \right) \right] , \tag{4.139a}$$

$$\Gamma_{m,2}(x) = \left[\lambda_2 \left(x + \frac{m-1}{2}\mathrm{i} \right) + \lambda_3 \left(x + \frac{m-1}{2}\mathrm{i} \right) \right] \prod_{k=2}^{m} \lambda_3 \left(x - \frac{2k-m-1}{2}\mathrm{i} \right) . \tag{4.139b}$$

It follows, that the uppercase auxiliary functions $B_{m,1}^{(1)}(x) = b_{m,1}^{(1)}(x) + 1$ are

$$B_{m,1}^{(1)}(x) = \frac{\Lambda_m^{(1)}(x + \frac{\mathrm{i}}{2})}{\Gamma_{m,2}(x + \frac{\mathrm{i}}{2})} , \qquad B_{m,2}^{(1)}(x) = \frac{\Lambda_m^{(1)}(x - \frac{\mathrm{i}}{2})}{\Gamma_{m,1}(x - \frac{\mathrm{i}}{2})} . \tag{4.140}$$

Analyzing the structure of the functions $\Gamma_{m,j}(x)$, we find

$$\Gamma_{m,1}(x) = n_m^{(1)}(x) \cdot \frac{\phi_+(x - \frac{m-1}{2}\mathrm{i})q_1(x + \frac{m+1}{2}\mathrm{i})q_1^{(h)}(x - \frac{m-1}{2}\mathrm{i})}{q_2(x - \frac{m-1}{2}\mathrm{i})} \cdot \mathrm{e}^{(m-1)\beta\mu_1}(\mathrm{e}^{\beta\mu_1} + \mathrm{e}^{\beta\mu_2}) , \tag{4.141a}$$

$$\Gamma_{m,2}(x) = n_m^{(1)}(x) \cdot \frac{\phi_-(x + \frac{m-1}{2}\mathrm{i})q_2(x - \frac{m+1}{2}\mathrm{i})q_2^{(h)}(x + \frac{m-1}{2}\mathrm{i})}{q_1(x + \frac{m-1}{2}\mathrm{i})} \cdot \mathrm{e}^{(m-1)\beta\mu_3}(\mathrm{e}^{\beta\mu_2} + \mathrm{e}^{\beta\mu_3}) \,. \tag{4.141b}$$

Then it is easy to write down all auxiliary functions in the factorized form. We get

$$b_{m,1}^{(1)}(x) = \frac{q_1(x + \frac{m+2}{2}\mathrm{i})\widetilde{\Lambda}_{m-1}^{(1)}(x)}{\phi_-(x + \frac{m}{2}\mathrm{i})q_2(x - \frac{m}{2}\mathrm{i})q_2^{(h)}(x + \frac{m}{2}\mathrm{i})} \cdot \frac{\mathrm{e}^{\beta\mu_1}a_{m-1}^{(1)}}{\mathrm{e}^{(m-1)\beta\mu_3}(\mathrm{e}^{\beta\mu_2} + \mathrm{e}^{\beta\mu_3})} \,, \tag{4.142a}$$

$$b_{m,1}^{(2)}(x) = \frac{q_2(x - \frac{m+2}{2}\mathrm{i})\widetilde{\Lambda}_{m-1}^{(1)}(x)}{\phi_+(x - \frac{m}{2}\mathrm{i})q_1(x + \frac{m}{2}\mathrm{i})q_1^{(h)}(x - \frac{m}{2}\mathrm{i})} \cdot \frac{\mathrm{e}^{\beta\mu_2}a_{m-1}^{(1)}}{\mathrm{e}^{(m-1)\beta\mu_1}(\mathrm{e}^{\beta\mu_1} + \mathrm{e}^{\beta\mu_2})} \tag{4.142b}$$

and

$$B_{m,1}^{(1)}(x) = \frac{q_1(x + \frac{m}{2}\mathrm{i})\widetilde{\Lambda}_{m}^{(1)}(x + \frac{\mathrm{i}}{2})}{\phi_-(x + \frac{m}{2}\mathrm{i})q_2(x - \frac{m}{2}\mathrm{i})q_2^{(h)}(x + \frac{m}{2}\mathrm{i})} \cdot \frac{a_{m}^{(1)}}{\mathrm{e}^{(m-1)\beta\mu_3}(\mathrm{e}^{\beta\mu_2} + \mathrm{e}^{\beta\mu_3})} \,, \tag{4.143a}$$

$$B_{m,1}^{(2)}(x) = \frac{q_2(x - \frac{m}{2}\mathrm{i})\widetilde{\Lambda}_{m}^{(1)}(x - \frac{\mathrm{i}}{2})}{\phi_+(x - \frac{m}{2}\mathrm{i})q_1(x + \frac{m}{2}\mathrm{i})q_1^{(h)}(x - \frac{m}{2}\mathrm{i})} \cdot \frac{a_{m}^{(1)}}{\mathrm{e}^{(m-1)\beta\mu_1}(\mathrm{e}^{\beta\mu_1} + \mathrm{e}^{\beta\mu_2})} \,. \tag{4.143b}$$

Like in the $sl(2)$-symmetric case, the connection to the original TBA auxiliary function $Y_m^{(1)}(x)$ is

$$Y_m^{(1)}(x) = B_{m,1}^{(1)}(x)B_{m,2}^{(1)}(x) \,, \tag{4.144}$$

and there exists the additional relation

$$b_{m,1}^{(1)}(x - \mathrm{i}/2)b_{m,2}^{(1)}(x + \mathrm{i}/2) = Y_{m-1}^{(1)}(x) \,. \tag{4.145}$$

Using the auxiliary functions $y_j^{(1)}(x)$ (for $j = 1, \ldots, m-1$), $y_1^{(2)}(x)$ and the two new functions $b_{m,1}^{(1)}(x)$ and $b_{m,2}^{(1)}(x)$, we can derive a closed set of NLIEs in the usual straightforward way. We find the result

$$\begin{aligned} \ln y_1^{(1)}(x) &= -\beta(JW^{(1)}(x) + d_1) + \left[K_0 * \ln Y_1^{(1)}\right](x) - \left[W^{(1)} * \ln Y_1^{(2)}\right](x) \\ &\quad + \left[V_{[2]}^{(1)} * \ln Y_2^{(1)}\right](x) \,, \end{aligned} \tag{4.146a}$$

$$\ln y_1^{(2)}(x) = -\beta(JW^{(2)}(x) + d_2) - \sum_{a=1}^{2} \left[W^{(a)} * \ln Y_1^{(a)}\right](x) \,, \tag{4.146b}$$

$$\ln y_k^{(1)}(x) = \left[V_{[2]}^{(1)} * \left(\ln Y_{k-1}^{(1)} + \ln Y_{k+1}^{(1)}\right)\right](x) \qquad (k = 2, \ldots, m-1) \,, \tag{4.146c}$$

$$\ln b_{m,j}^{(1)}(x) = -\beta c_j + \left[V_{[2]}^{(1)} * \ln Y_{m-1}^{(1)}\right](x) - \sum_{k=1}^{2} \left[\mathbf{K}_{j,k}^{(1,1)} * \ln B_{m,k}^{(1)}\right](x) \,, \tag{4.146d}$$

where the kernel function $K_0(x)$ and the kernel matrix $\mathbf{K}^{(1,1)}(x)$ are those of the $sl(2)$-symmetric case, see equations (4.10) and (4.123). The functions $W^{(a)}(x)$ are defined as

$$W^{(a)}(x) = \frac{4a}{4x^2 + a^2}\,, \tag{4.147}$$

and the constants are found to be

$$c_1 = \frac{\mu_3 - \mu_1}{2}\,, \qquad c_2 = \frac{\mu_1 - \mu_3}{2}\,, \tag{4.148a}$$

$$d_1 = \frac{2\mu_2 - \mu_1 - \mu_3}{2}\,, \qquad d_2 = 2\mu_2 - \mu_1 - \mu_3\,. \tag{4.148b}$$

The final expression for the eigenvalue is

$$\begin{aligned} \ln \Lambda_1^{(1)}(0) &= \beta(J + \mu_2) + \sum_{a=1}^{2} \left[W^{(a)} * \ln Y_1^{(a)} \right](0) \\ &= -\beta\,(J + \mu_2 - \mu_1 - \mu_3) - \ln y_1^{(2)}(0)\,. \end{aligned} \tag{4.149}$$

We like to stress the strong resemblance of the set of NLIEs (4.146) to the corresponding equations (4.121) and (4.122) of the $sl(2)$-symmetric case. Despite the differences for the auxiliary functions $\ln y_1^{(a)}(x)$, the structure is exactly the same. This result is expected, nevertheless, if we recall the mutual similarity between the TBA equations of both the general $sl(n)$- and $sl(n|1)$-symmetric cases that we have pointed out in Section 3.4.

Chapter 5

Conjectures for higher ranks

The derivation of the finite coupled sets of NLIEs presented in the last chapter is based on the knowledge of suitable auxiliary functions. Apart from the assumption of certain analyticity properties, which are backed by numerics, it is completely rigorous. Unfortunately, no method to generally construct these functions is known so far, so that a considerable amount of trial and error is necessary to find the proper auxiliary functions for each case. On the background of a rising number of such functions with increasing complexity, the derivation of NLIEs for higher-rank cases of the Uimin-Sutherland model seems to be out of reach. However, based on the known results it is well possible to conjecture much of the general structure of the final NLIEs. This will be enough to propose the NLIEs for the $sl(5)$- and $sl(4|1)$-symmetric cases of the Uimin-Sutherland model. Note that these conjectures are backed by the numerical results of the next chapter.

5.1 General structure for the $sl(n)$-symmetric case

For the $sl(n)$-symmetric case of the Uimin-Sutherland model, we expect to have as many sets of auxiliary functions as is the rank of the underlying algebra, while the number of auxiliary functions within each set a should match the dimension of the ath fundamental representation $d_a = \binom{n}{a}$. This gives a total number of $N = 2^n - 2$ many functions. Moreover, the uppercase forms of the auxiliary functions are expected to be connected to the auxiliary functions of the TBA approach via

$$Y_1^{(a)}(x) = \prod_{j=1}^{d_a} B_{1,j}^{(a)}(x) \,. \tag{5.1}$$

The final NLIEs should be of convolution type, where the structure is

$$\ln b_{1,j}^{(a)}(x) = -\beta \left(J V_{[n]}^{(a)}(x) + c_j^{(a)} \right) - \sum_{b=1}^{n-1} \sum_{k=1}^{d_b} \left[\mathbf{K}_{j,k}^{(a,b)} * \ln B_{1,k}^{(b)} \right](x) \,. \tag{5.2}$$

For the largest eigenvalue of the QTM, we get the equation

$$\ln \Lambda_1^{(1)}(0) = -\beta \left\{ J \left[1 - \frac{2}{n} \left(\psi(1) - \psi\left(\frac{1}{n}\right) \right) \right] - \frac{1}{n} \sum_{j=1}^{n} \mu_j \right\} + \sum_{a=1}^{n-1} \sum_{j=1}^{d_b} \left[V_{[n]}^{(a)} * \ln B_{1,j}^{(a)} \right](0)\,, \tag{5.3}$$

which is equivalent to the corresponding eigenvalue formula known from the TBA approach, see equation (3.33). The function $V_{[n]}^{(a)}(x)$ is given by

$$V_{[n]}^{(a)}(x) = \frac{2\pi}{n} \frac{\sin(\pi a/n)}{\cosh(2\pi x/n) - \cos(\pi a/n)}\,, \tag{5.4}$$

confer equation (4.56).

In order to conjecture the explicit form of the constants $c_j^{(a)}$, recall that at the end of Section 4.3.1 it proved useful to assign Young tableaux to the auxiliary functions. These Young tableaux are also helpful here. Let us denote by $Y(a,j)$ the set of numbers that appear in the Young tableau assigned to the jth auxiliary function of the ath set. Then, we notice that the constants $c_j^{(a)}$ of all previously derived NLIEs of $sl(n)$ type, confer equations (4.12), (4.25) and (4.63), are compatible with the general form

$$c_j^{(a)} = \frac{a}{n} \sum_{k=1}^{n} \mu_k - \sum_{l \in Y(a,j)} \mu_l\,. \tag{5.5}$$

Having fixed the constants, only the kernel matrices $\mathbf{K}^{(a,b)}(x)$ yet remain to be determined. Note that, as usual, we assume the matrices $\mathbf{K}^{(a,b)}(x)$ to be the submatrices of one big matrix

$$\mathbf{K}(x) = \begin{pmatrix} \mathbf{K}^{(1,1)}(x) & \cdots & \mathbf{K}^{(1,n-1)}(x) \\ \vdots & \ddots & \vdots \\ \mathbf{K}^{(n-1,1)}(x) & \cdots & \mathbf{K}^{(n-1,n-1)}(x) \end{pmatrix}, \tag{5.6}$$

which we expect to be both Hermitian and invariant under a reflection along the antidiagonal,

$$\mathbf{K}(x) = (\mathbf{K}(x))^\dagger\,, \qquad K_{j,k}(x) = K_{N+1-k,N+1-j}(x)\,. \tag{5.7}$$

Remarkably, it turns out that most of the entries of this matrix can be deduced by investigating certain limiting cases of the NLIEs (5.2).

Let us start with the zero-temperature limit and recall the results of Section 4.3.2. We have already seen that the number of contributing auxiliary functions reduces to $n-1$ in this limit. The corresponding selection rule depends on the actual ordering of the general chemical potentials, $\mu_{P(1)} \geq \mu_{P(2)} \geq \ldots \geq \mu_{P(n)}$, where $P(j)$ is some fixed permutation of the indices $1,\ldots,n$. The resulting linearized integral equations, nevertheless, are structurally the same for all $n!$ possible permutations and identical to the corresponding zero-temperature TBA equations (3.43). Only the constants $c^{(a)}$ depend on the actual choice of $P(j)$. Since all kernel functions $\mathcal{K}_{[n]}^{(a,b)}(x)$ of the zero-temperature TBA equations are explicitly known, see equation (3.44), we can now use the selection rules to deduce their original positions within $\mathbf{K}(x)$. Let us formulate the result with the help of the sets $Y(a,j)$. We find that $\mathbf{K}_{j,k}^{(a,b)}(x) = \mathcal{K}_{[n]}^{(a,b)}(x)$ if $Y(a,j) \subseteq Y(b,k)$ or $Y(b,k) \subseteq Y(a,j)$.

Next, we consider the limiting case where one basis state α is frozen out by sending the corresponding chemical potential μ_α to minus infinity. As we already know from Section 4.3.1, the NLIEs (5.2) of the $sl(n)$-symmetric case are expected to reduce to those of the $sl(n-1)$-symmetric case in this limit. Moreover, the conjectured selection rule says that only those auxiliary functions $b_{1,j}^{(a)}(x)$ will survive in this process, for which $\alpha \notin Y(a,j)$. The single auxiliary function $b_{1,s}^{(n-1)}(x)$ that survives from the last set will diverge and the corresponding NLIE linearizes, since $b_{1,s}^{(n-1)}(x)/B_{1,s}^{(n-1)}(x) \to 1$. Note that we already know all kernel functions that appear in this equation, because $Y(a,j) \subseteq Y(n-1,s)$ for all surviving functions $B_{1,j}^{(a)}(x)$. Therefore, we get the result

$$\ln B_{1,s}^{(n-1)}(x) = -\beta\left(JV_{[n]}^{(n-1)}(x) + c_s^{(n-1)}\right) - \sum_{b=1}^{n-1} \sum_{\substack{k=1 \\ \alpha\notin Y(b,k)}}^{d_b} \left[\mathcal{K}_{[n]}^{(n-1,b)} * \ln B_{1,k}^{(b)}\right](x)\,. \tag{5.8}$$

With the help of the transformation (4.4), this equation can now be solved with respect to $\ln B_{1,s}^{(n-1)}(x)$. Substituting the result into the equations for the other nonvanishing functions $b_{1,j}^{(a)}(x)$, we finally get the result

$$\ln b_{1,j}^{(a)}(x) = -\beta\left(JV_{[n-1]}^{(a)}(x) + d_j^{(a)}\right) - \sum_{b=1}^{n-2} \sum_{\substack{k=1 \\ \alpha\notin Y(b,k)}}^{d_b} \left[\mathbf{L}_{j,k}^{(a,b)} * \ln B_{1,k}^{(b)}\right](x)\,, \tag{5.9}$$

where

$$\mathbf{L}_{j,k}^{(a,b)}(x) = \mathbf{K}_{j,k}^{(a,b)}(x) - \int_{-\infty}^{\infty} \mathrm{e}^{|k|/2} \frac{\sinh(ak/2)\sinh(bk/2)}{\sinh((n-1)k/2)\sinh(nk/2)}\, \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}x\,. \tag{5.10}$$

After relabelling the $2^{n-1}-2$ many surviving functions $b_{1,j}^{(a)}(x)$ in order to fill the gaps in the index nomenclature, the set of NLIEs given by (5.9) is indeed of the form that we expect for the $sl(n-1)$-symmetric case.

Suppose now that the NLIEs of the $sl(n-1)$-symmetric case are already known. Then we also know all functions $\mathbf{L}_{j,k}^{(a,b)}(x)$—and therefore the sought-after kernel functions $\mathbf{K}_{j,k}^{(a,b)}(x)$ of the $sl(n)$-symmetric case—for which $\alpha \notin Y(a,j)$ and $\alpha \notin Y(b,k)$. Note that we expect all kernel functions to be of the form

$$\mathbf{K}_{j,k}^{(a,b)}(x) = \mathcal{K}_{[n]}^{(a,b)}(x) + R_{j,k}^{(a,b)}(x)\,, \tag{5.11}$$

where $R_{j,k}^{(a,b)}(x)$ is a rational function. Remarkably, the rational part $R_{j,k}^{(a,b)}(x)$ must then be the same for both $\mathbf{K}_{j,k}^{(a,b)}(x)$ and $\mathbf{L}_{j,k}^{(a,b)}(x)$, because of the identity

$$\mathcal{K}_{[n-1]}^{(a,b)}(x) = \mathcal{K}_{[n]}^{(a,b)}(x) - \int_{-\infty}^{\infty} \mathrm{e}^{|k|/2} \frac{\sinh(ak/2)\sinh(bk/2)}{\sinh((n-1)k/2)\sinh(nk/2)}\, \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}x\,. \tag{5.12}$$

Since we are free to choose any basis state $\alpha \in \{1,\ldots,n\}$ to perform the limit, we can eventually get all kernel functions $\mathbf{K}^{(a,b)}_{j,k}(x)$ for which α exists such that $\alpha \notin Y(a,j)$ and $\alpha \notin Y(b,k)$. This will definitely not be the case if $Y(a,j) \cup Y(b,k) = \{1,\ldots,n\}$, which may happen in the lower right triangle of the kernel matrix $\mathbf{K}(x)$. Nevertheless, these kernel functions can still be gained from the upper left triangle due to the fact that $\mathbf{K}(x)$ should be symmetric along the antidiagonal. Note, however, that we have a problem if the sets $Y(a,j)$ and $Y(b,k)$ turn out to be disjunct. This can only happen if $a+b=n$ and $j+k=d_a+1$, that is exactly on the antidiagonal of the kernel matrix $\mathbf{K}(x)$. These kernel functions remain unknown. Still, it is a quite remarkable result that nearly all kernel functions of the $sl(n)$-symmetric case follow directly from those of the $sl(n-1)$-symmetric case.

Although the above procedure tells us nothing on the kernel functions that are still missing, we have actually found a way to guess these functions on the basis of certain additional algebra-related properties of the kernel matrix $\mathbf{K}(x)$. The method is briefly explained in Appendix D. Unfortunately, the additional structure is not yet well understood.

Based on the known results for the $sl(4)$-symmetric case, we are finally able to conjecture the complete kernel matrix of the $sl(5)$-symmetric case, where we have 30 auxiliary functions, divided into sets of 5, 10, 10 and 5 many functions. The explicit result is given in Appendix E.

5.2 General structure for the $sl(n|1)$-symmetric case

Let us now turn to the general $sl(n|1)$-symmetric case of the Uimin-Sutherland model, for which the NLIEs can be conjectured in an analogous way. Here, the total number of auxiliary functions is expected to be $N = 2^n - 1$, where the functions are dividable into n sets of $d_a = \binom{n}{a}$ many auxiliary functions. The connection to the TBA auxiliary functions $Y_1^{(a)}(x)$ is then given by

$$Y_1^{(a)}(x) = \prod_{j=1}^{d_a} B_{1,j}^{(a)}(x)\,. \tag{5.13}$$

Note that the last set $(a=n)$ contains only one function which therefore must be identical to the corresponding TBA auxiliary function, $y_1^{(n)}(x) = b_{1,1}^{(n)}(x)$. Based on the results of the previous chapter, the general form of the coupled NLIEs is expected to be

$$\ln b_{1,j}^{(a)}(x) = -\beta\left(JV^{(a)}(x) + c_j^{(a)}\right) - \sum_{b=1}^{n}\sum_{k=1}^{d_b}\left[\mathbf{K}^{(a,b)}_{j,k} * \ln B_{1,j}^{(a)}\right](x)\,, \tag{5.14}$$

while the expression for the largest eigenvalue is already known from the TBA calculations,

$$\ln \Lambda_1^{(1)}(0) = \beta(J+\mu_g) + \sum_{a=1}^{n}\sum_{j=1}^{d_a}\left[V^{(a)} * \ln B_{1,j}^{(a)}\right](0)\,, \tag{5.15}$$

confer equation (3.40), where g is the graded basis state for which $p(g)=1$. We find the functions $V^{(a)}(x)$ to be

$$V^{(a)}(x) = \frac{4a}{4x^2+a^2}\,, \tag{5.16}$$

and the constants $c_j^{(a)}$ are

$$c_j^{(a)} = a\mu_g - \sum_{k\in Y(a,j)} \mu_j \,, \tag{5.17}$$

where we have again used the sets of numbers $Y(a,j)$ appearing in the Young tableaux, which we have assigned to the auxiliary functions in Section 4.3.1.

The complete kernel matrix $\mathbf{K}(x)$ will be Hermitian, but we do no longer expect it to be invariant under reflection along the antidiagonal. In order to get some explicit information on the kernel functions $\mathbf{K}_{j,k}^{(a,b)}(x)$, let us analyze the zero-temperature limit of the NLIEs again, where the resulting linearized integral equations have to coincide with the corresponding zero-temperature TBA equations, see equation (3.46). We learn that $\mathbf{K}_{j,k}^{(a,b)}(x) = \mathcal{K}^{(a,b)}(x)$ if $Y(a,j) \subseteq Y(b,k)$ or $Y(b,k) \subseteq Y(a,j)$, where

$$\mathcal{K}^{(a,b)}(x) = \int_{-\infty}^{\infty} \left\{ \mathrm{e}^{-(\max(a,b)-1)|k|/2} \frac{\sinh(\min(a,b)k/2)}{\sinh(k/2)} - \delta_{a,b} \right\} \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}k \,. \tag{5.18}$$

This particularly implies that all kernel functions $\mathbf{K}_{1,j}^{(n,a)}(x) = \mathbf{K}_{j,1}^{(a,n)}(x)$ are explicitly known.

In order to determine the remaining entries of the kernel matrices $\mathbf{K}^{(a,b)}(x)$ with $a, b < n$, suppose now that we already know the results for the $sl(n)$-symmetric case and recall the final result of Section 4.1.2, where we have found a close connection between the kernel matrix of the $sl(3|1)$- and that of the $sl(3)$-symmetric case. Admitting this structure in general, we can make the following conjecture: Suppose that $R_{j,k}^{(a,b)}(x)$ is the rational part of some kernel function $\mathbf{K}_{j,k}^{(a,b)}(x)$ of the $sl(n)$-symmetric case. Then the corresponding kernel function $\mathbf{K}_{j,k}^{(a,b)}(x)$ of the $sl(n|1)$-symmetric case will be given by

$$\mathbf{K}_{j,k}^{(a,b)} = \mathcal{K}^{(a,b)}(x) + R_{j,k}^{(a,b)}(x) \,. \tag{5.19}$$

Based on the explicit results that we have obtained for the $sl(4)$-symmetric case in Section 4.2.1, for example, we can now easily get the complete 15 by 15 kernel matrix $\mathbf{K}(x)$ of the $sl(4|1)$-symmetric case. Note that the 31 by 31 kernel matrix of the $sl(5|1)$-symmetric case similarly follows from our conjecture for the $sl(5)$-symmetric case.

5.3 Exact truncation of the TBA equations for $sl(n)$ and $sl(n|1)$

In Section 4.4 we have already seen that, after some slight modification of the auxiliary functions $b_{1,j}^{(a)}(x)$, one can find new functions $b_{m,j}^{(a)}(x)$ such that

$$Y_m^{(a)}(x) = \prod_{j=1}^{d_a} B_{m,j}^{(a)}(x) \,. \tag{5.20}$$

Moreover, we have seen that these functions can be used to exactly truncate the set of TBA equations $y_k^{(a)}(x)$ at some arbitrary level $k = m$. Based on the structure we found there, we are able to conjecture the final form of the truncated TBA equations for both the general $sl(n)$- and $sl(n|1)$-symmetric cases.

For the $sl(n)$-symmetric case, we expect the result to be

$$\begin{aligned}\ln y_k^{(a)}(x) = -J\beta\delta_{k,1}V_{[n]}^{(a)}(x) + \sum_{b=1}^{n-1}\Big\{ \Big[A_{[n]}^{(a,b)} * \Big((1-\delta_{m,1})\ln Y_{k-1}^{(b)} + \ln Y_{k+1}^{(b)}\Big)\Big](x) \\ - \Big[\big(A_{[n]}^{(a,b-1)} + A_{[n]}^{(a,b+1)}\big) * \ln Y_k^{(b)}\Big](x)\Big\}\,, \end{aligned} \tag{5.21a}$$

$$\ln b_{m,j}^{(a)}(x) = -\beta c_j^{(a)} + \sum_{b=1}^{n-1}\left\{\Big[A_{[n]}^{(a,b)} * \ln Y_{m-1}^{(b)}\Big](x) - \sum_{k=1}^{d_b}\Big[\mathbf{K}_{j,k}^{(a,b)} * \ln B_{m.k}^{(b)}\Big](x)\right\}. \tag{5.21b}$$

Note that the functions $A_{[n]}^{(a,b)}(x)$ are exactly those from the usual TBA equations (3.25), while the functions $V_{[n]}^{(a)}(x)$, constants $c_j^{(a)}$ and kernel matrices $\mathbf{K}^{(a,b)}(x)$ are those from the NLIEs (5.2).

In order to deal with the $sl(n|1)$-symmetric case, we need to modify the equations (5.21) only slightly, since the TBA equations for both cases are closely related. Instead of the equations for $\ln y_1^{(a)}(x)$, we will now have the equations

$$\begin{aligned}\ln y_1^{(a)}(x) = -\beta\left(J\frac{4a}{4x^2+a^2} + d^{(a)}\right) + \sum_{b=1}^{n-1}\Big[A_{[n]}^{(a,b)} * \ln Y_2^{(b)}\Big](x) \\ - \sum_{b=1}^{n}\Big[\Big(A_{[n]}^{(a,b-1)} + A_{[n]}^{(a,b+1)} + C^{(a,b)}\Big) * \ln Y_1^{(b)}\Big](x)\,, \end{aligned} \tag{5.22a}$$

$$\ln y_1^{(n)}(x) = -\beta\left(J\frac{4n}{4x^2+n^2} + d^{(n)}\right) - \sum_{b=1}^{n}\Big[C^{(n,b)} * \ln Y_1^{(b)}\Big](x) - \ln Y_1^{(n)}(x)\,. \tag{5.22b}$$

The constants $d^{(a)}$ are those given in equation (3.38), and the functions $C^{(a,b)}(x)$ have been defined in equation (3.39). Note that we also have to change the constants $c_j^{(a)}$. Suppose again that g is the label of the basis state with grading $p(g) = 1$. Then all indices j of the general chemical potentials μ_j are shifted by one, $j \mapsto j+1$, if $j \geq g$.

Chapter 6

Numerical investigation

The finite sets of coupled nonlinear integral equations (NLIEs) that have been obtained in the previous two chapters are well suited for a numerical evaluation at arbitrary finite temperature and chemical potentials. In this chapter, we will briefly explain the necessary methods and show, how the results can be used to derive various thermodynamical properties.

6.1 Numerical treatment of the NLIEs

Since our sets of NLIEs are self-consistent, it appears promising to strive for a solution by iteration. We start with some initial approximation for the set of functions $\ln b^{(a)}_{m,j}(x)$ and compute the functions $\ln B^{(a)}_{m,j}(x)$ and the convolutions on the right hand sides of the NLIEs to eventually obtain a new approximation for the set of functions $\ln b^{(a)}_{m,j}(x)$. Then, we restart the cycle with this new approximation. These steps have to be repeated until all auxiliary functions have converged within some given error margin. In the end, the largest eigenvalue of the QTM can be calculated from the known set of functions $\ln B^{(a)}_{m,j}(x)$.

In order to do the calculations on a computer, it is of course necessary to clip and discretize all involved functions. We achieve this by sampling all functions on the real axis at a fixed number of equally distributed points within a fixed interval around the origin. The convolutions on the right hand sides of the NLIEs are best evaluated in Fourier space, since the Fourier transformation turns them into simple multiplications. The necessary transformations can efficiently be calculated by use of the fast Fourier transform (FFT) algorithm. We like to stress that the truncation and discretization of the functions are the only sources of systematic error within this approach. In comparison to these, the additional numerical roundoff error is generally found to be negligible. Fortunately, the systematic error can easily be controlled by comparing the results gained for different choices of the interval and the number of sampling points. Note that the knowledge of a reliable error margin for all numerical results is a big advantage of the current approach.

The largest eigenvalue of the QTM is directly related to the free energy of the model.

Since we are also interested in certain derivatives of the free energy, we additionally need to calculate the corresponding derivatives of the largest eigenvalue. Instead of dealing with potentially ill-conditioned numerical derivatives, however, it is possible to pursue an alternative approach where we directly work with derivatives of the equation for the eigenvalue and the NLIEs. It uses the fact, that the derivative of each uppercase function $\ln B^{(a)}_{m,j}(x)$ with respect to some parameter p is related to the derivative of the corresponding lowercase function $\ln b^{(a)}_{m,j}(x)$ by

$$\frac{\partial}{\partial p} \ln B^{(a)}_{m,j}(x) = \frac{b^{(a)}_{m,j}(x)}{B^{(a)}_{m,j}(x)} \cdot \frac{\partial}{\partial p} \ln b^{(a)}_{m,j}(x) \,. \tag{6.1}$$

Thus, the differentiated NLIEs can be solved in the same iterative fashion as the original NLIEs, once the results for all functions $\ln b^{(a)}_{m,j}(x)$ are known. Similarly, it is possible to treat higher derivatives. For the second derivative with respect to p, for example, we find the relation

$$\frac{\partial^2}{\partial p^2} \ln B^{(a)}_{m,j}(x) = \frac{b^{(a)}_{m,j}(x)}{B^{(a)}_{m,j}(x)} \left\{ \frac{1}{B^{(a)}_{m,j}(x)} \left(\frac{\partial}{\partial p} \ln b^{(a)}_{m,j}(x) \right)^2 + \frac{\partial^2}{\partial p^2} \ln b^{(a)}_{m,j}(x) \right\} , \tag{6.2}$$

which gives the relation between $\frac{\partial^2}{\partial p^2} \ln b^{(a)}_{m,j}(x)$ and $\frac{\partial^2}{\partial p^2} \ln B^{(a)}_{m,j}(x)$. Since here the functions $\ln b^{(a)}_{m,j}(x)$ and $\frac{\partial}{\partial p} \ln b^{(a)}_{m,j}(x)$ appear, these have to be calculated first.

Note that the above procedure does not introduce any significant additional numerical errors, so that we can get reliable error estimates even for the derived properties.

6.2 Calculation of thermodynamical properties

In the thermodynamic limit, the free energy f of the Uimin-Sutherland model is related to the largest eigenvalue of the QTM via

$$f = -\frac{1}{\beta} \ln \Lambda^{(1)}_1(0) \,. \tag{6.3}$$

Moreover, we are interested in the entropy S, specific heat C, particle density n and the compressibility κ, which are defined by

$$S = -\left(\frac{\partial f}{\partial T}\right)_\mu , \qquad C = T \left(\frac{\partial S}{\partial T}\right)_n , \qquad n = -\left(\frac{\partial f}{\partial \mu}\right)_T , \qquad \kappa = \left(\frac{\partial n}{\partial \mu}\right)_T . \tag{6.4}$$

Note that since we are working in the grand canonical formulation with a fixed chemical potential μ, we have to use the relation

$$\left(\frac{\partial S}{\partial T}\right)_n = \left(\frac{\partial S}{\partial T}\right)_\mu - \left(\frac{\partial n}{\partial T}\right)^2_\mu \left(\frac{\partial n}{\partial \mu}\right)^{-1}_T \tag{6.5}$$

in order to actually calculate the specific heat. If some external magnetic field h is applied, we are additionally interested in the magnetization M and the magnetic susceptibility χ defined by

$$M = -\frac{\partial f}{\partial h}\,, \qquad\qquad \chi = \frac{\partial M}{\partial h}\,. \tag{6.6}$$

With the help of equation (6.3), it is easy to see that all the properties can be written solely using derivatives of the largest eigenvalue,

$$S = \mathrm{l}\Lambda - \beta\frac{\partial}{\partial\beta}\mathrm{l}\Lambda\,, \qquad C = \beta^2\frac{\partial^2}{\partial\beta^2}\mathrm{l}\Lambda - \frac{\left(\frac{\partial}{\partial\mu}\mathrm{l}\Lambda - \beta\frac{\partial^2}{\partial\beta\partial\mu}\mathrm{l}\Lambda\right)^2}{\frac{\partial^2}{\partial\mu^2}\mathrm{l}\Lambda}\,, \tag{6.7a}$$

$$n = \frac{1}{\beta}\cdot\frac{\partial}{\partial\mu}\mathrm{l}\Lambda\,, \qquad \kappa = \frac{1}{\beta}\cdot\frac{\partial^2}{\partial\mu^2}\mathrm{l}\Lambda\,, \tag{6.7b}$$

$$M = \frac{1}{\beta}\cdot\frac{\partial}{\partial h}\mathrm{l}\Lambda\,, \qquad \chi = \frac{1}{\beta}\cdot\frac{\partial^2}{\partial h^2}\mathrm{l}\Lambda\,, \tag{6.7c}$$

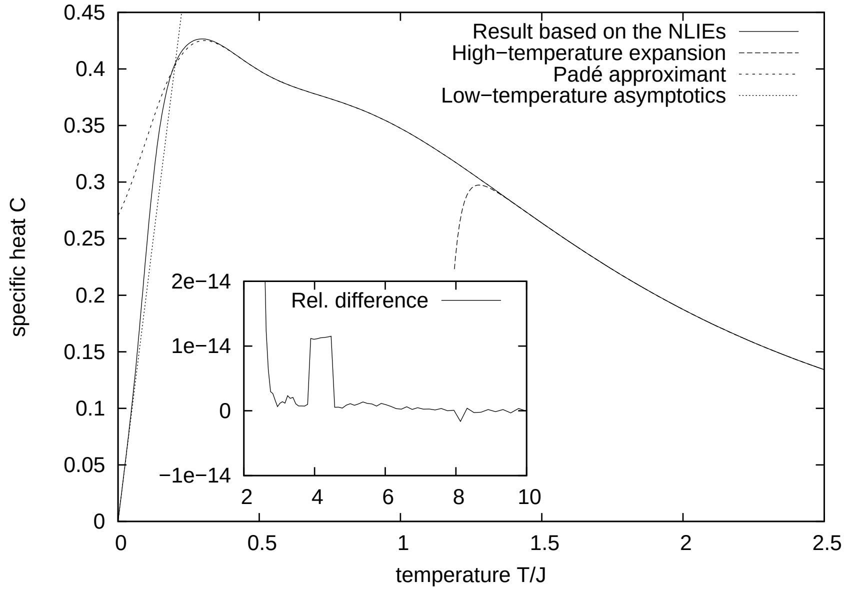
where $\mathrm{l}\Lambda$ is used as a shorthand for $\ln\Lambda_1^{(1)}(0)$. Therefore, it is possible to derive all the properties with good accuracy just by solving the corresponding derivatives of the NLIEs using the approach from the previous section.

6.3 Numerical results

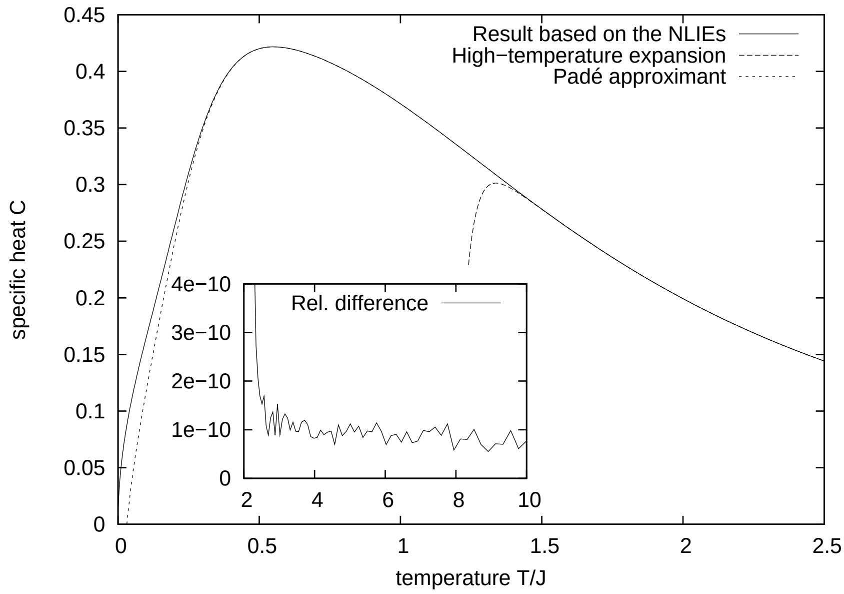
In the following, we start with a comparison of the results obtained by our NLIEs to existing high-temperature expansions. Thereafter, we give results for the various applications of the four- and five-state cases of the Uimin-Sutherland model that have been introduced in Section 2.2. Note that we will always assume $J > 0$. The models based on the two- and three-state cases are skipped, since their NLIEs have been previously known, and extensive numerical investigations for these models already exist in the literature, see for example [27, 37, 41].

6.3.1 Comparison with high-temperature expansions

We have checked the validity of our specific-heat results against existing high-temperature expansions, which have been derived by Tsuboi [91] up to very high orders (~ 40). Figures 6.1 to 6.3 show the comparison for the various special cases of the Uimin-Sutherland model. Since the high-temperature expansions naturally diverge above $T/J = 1.0$, we have also plotted the Padé approximants based on the expansions, which give reasonable results even for lower temperature. Obviously, we get an excellent agreement in all considered cases. Remarkably, the Padé approximants coincide with the results from our NLIEs down to $T/J \approx 0.3$. Only the results based on our NLIEs are also valid for arbitrary finite temperature and thus show the expected low-temperature behaviour, where the specific heat finally drops to zero. In the high-temperature regime, the accuracy of our results is generally found to be better

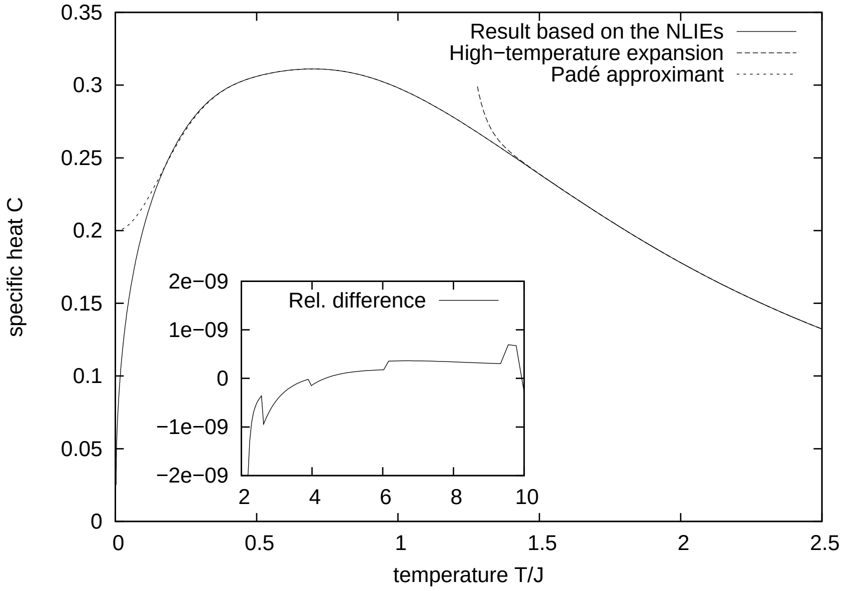


(a) Specific heat of the $sl(4)$-symmetric Uimin-Sutherland model.

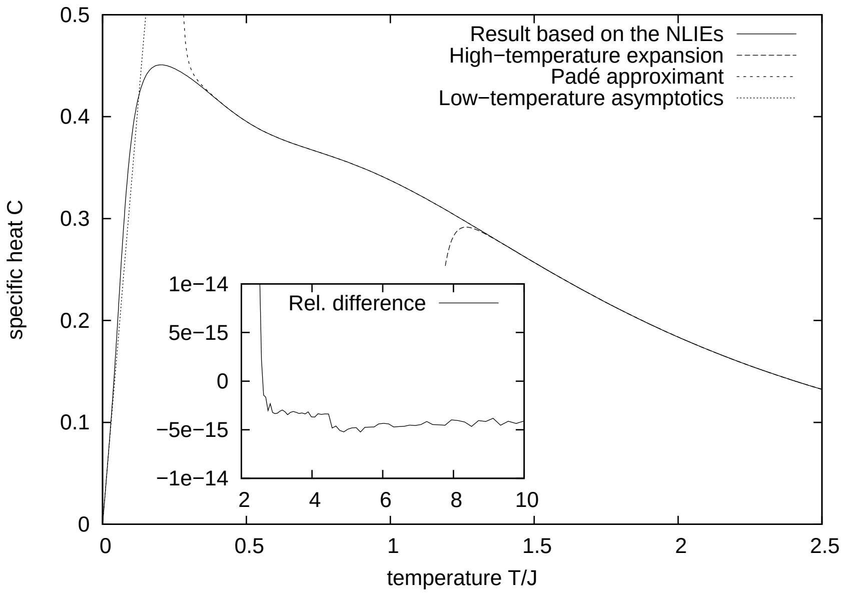


(b) Specific heat of the $sl(3|1)$-symmetric Uimin-Sutherland model.

Figure 6.1: Comparison of specific-heat results for the $sl(4)$- and $sl(3|1)$-symmetric cases of the Uimin-Sutherland model. The Padé approximants are based on the high-temperature expansions. The insets show the relative difference between the data from the NLIEs and the high-temperature expansion in the high-temperature regime.



(a) Specific heat of the $sl(2|2)$-symmetric Uimin-Sutherland model.



(b) Specific heat of the $sl(5)$-symmetric Uimin-Sutherland model.

Figure 6.2: Comparison of specific-heat results for the $sl(2|2)$- and $sl(5)$-symmetric cases of the Uimin-Sutherland model. The Padé approximants are based on the high-temperature expansions. The insets show the relative difference between the data from the NLIEs and the high-temperature expansion in the high-temperature regime.

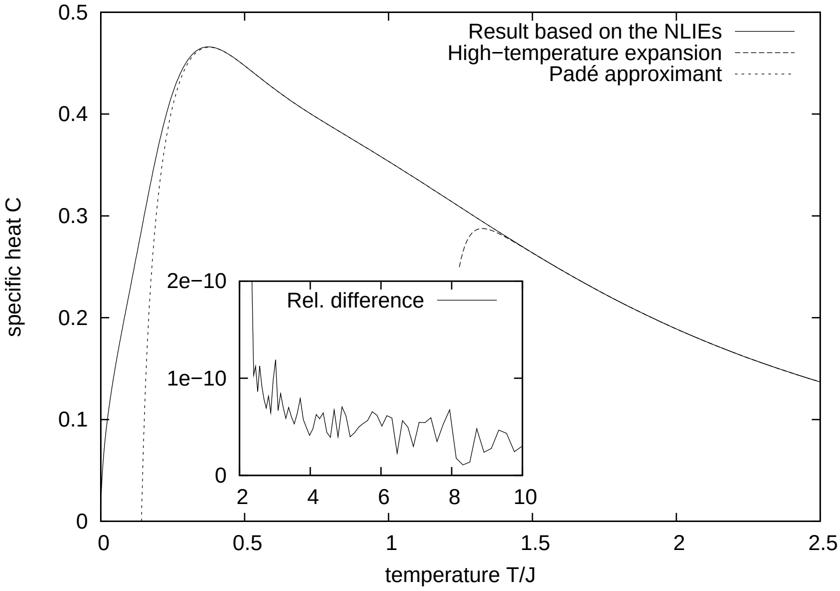


Figure 6.3: Comparison of specific-heat results for the $sl(4|1)$-symmetric case of the Uimin-Sutherland model. The Padé approximant is based on the high-temperature expansion. The inset shows the relative difference between the data from the NLIEs and the high-temperature expansion in the high-temperature regime.

than 10^{-9}. For the $sl(4)$- and $sl(5)$-symmetric cases, an even better accuracy around 10^{-14} is reached. Note that for these two cases the low-temperature asymptotics of the free energy follows from conformal field theory to be

$$f \simeq f_0 - \frac{\pi c}{6 v J} T^2 \, , \tag{6.8}$$

where c is the central charge and v is the sound velocity [2, 13]. For the $sl(4)$-symmetric case we have $c = 3$ and $v = \pi/2$, whereas for the $sl(5)$-symmetric case the values are $c = 4$ and $v = 2\pi/5$, confer [19, 74]. The low-temperature specific heat data are therefore expected to asymptotically yield $C \simeq 2T/J$ and $C \simeq 10T/(3J)$, respectively, in full agreement with our results. Note also that the enormous accuracy of our data is achieved only because of the vanishing general chemical potentials. In the presence of external fields, we typically find the numerical error to be in the order of 10^{-3} to 10^{-6}.

Finally, we like to stress that especially for the $sl(5)$- and $sl(4|1)$-symmetric cases, where the corresponding sets of NLIEs are only conjectures, these results can be taken as a strong evidence for the validity of the NLIEs and support our assumptions on the general structure.

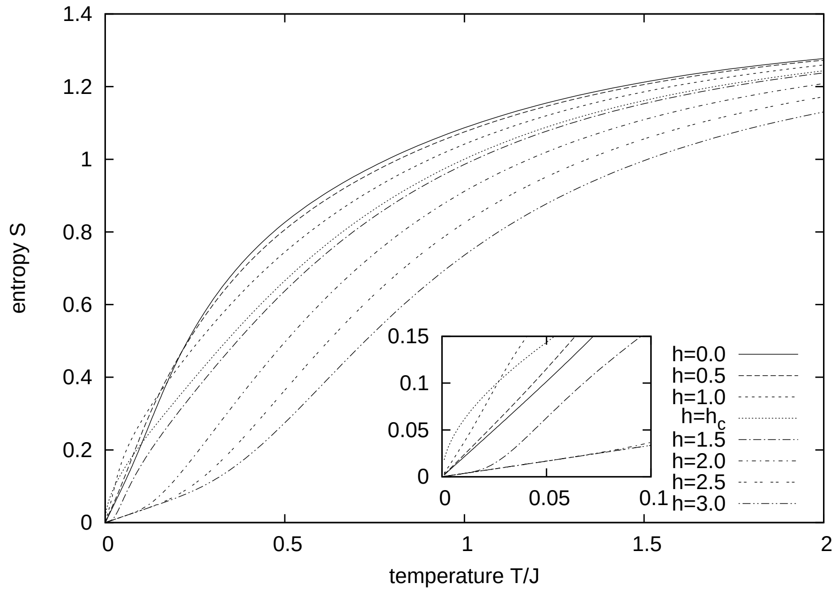
6.3.2 The $SU(4)$ spin-orbital model

Let us now turn to the $SU(4)$ spin-orbital model introduced in Section 2.2.4. Note that the thermodynamical properties of this model have already been studied numerically using various methods [26, 28, 32, 72, 91]. Still, none of these methods is capable of providing highly accurate results for low finite temperature in the thermodynamic limit.

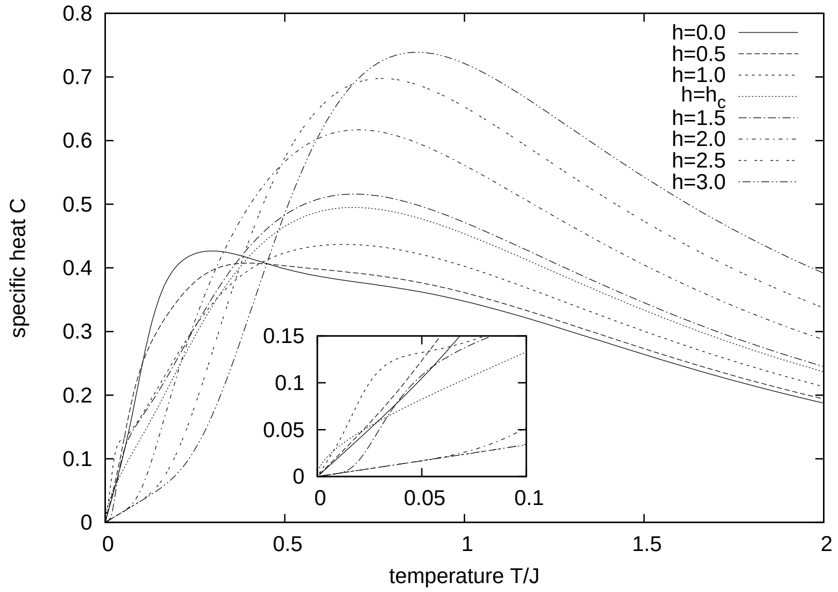
We first consider the special case, where the external magnetic field couples only to spins but not at all to the pseudospin degrees of freedom. Accordingly, the Landé factors are $g_S = 1$ and $g_\mathcal{T} = 0$ and we have $\mu_1 = \mu_2 = h/2$ and $\mu_3 = \mu_4 = -h/2$. From our analytical investigation of the zero-temperature limit in Section 4.3.2, we already know that there exists only one critical magnetic field in this case, which is given by $h_c = 2J \ln 2 \approx 1.39J$. If the external field is below this critical value, all four basis states contribute to the ground state. Above the critical field, the spins are fully polarized and only the orbital degrees of freedom remain.

The phase transition is clearly exposed by the numerical data given in Figures 6.4 and 6.5. The low-temperature slopes both of the entropy and the specific heat increase from 2 at $h = 0$ to infinity at $h = h_c$, whereas a constant value of 1/3 is obtained for $h > h_c$. Moreover, the magnetization data shows the expected saturation behaviour for $h \geq h_c$. Note also that the magnetic susceptibility diverges at the critical field. Below, the value at $T = 0$ stays finite; above, it drops to zero.

The magnetic susceptibility at $h = 0$ is particularly interesting, since it is expected to show a characteristic singular behaviour at $T = 0$ due to logarithmic corrections. Indeed, this is confirmed by our results for the low-temperature susceptibility, see Figure 6.6 on page 92. Even for the lowest plotted temperature, $T/J = 10^{-10}$, the susceptibility is still well above the ground-state value $\chi(0) \cdot J = 2/\pi^2$. For the spin-1/2 Heisenberg model, these corrections have already been treated in detail [22, 42, 45, 63]; similar results for the general $sl(n)$-symmetric Uimin-Sutherland model are known [27, 68].

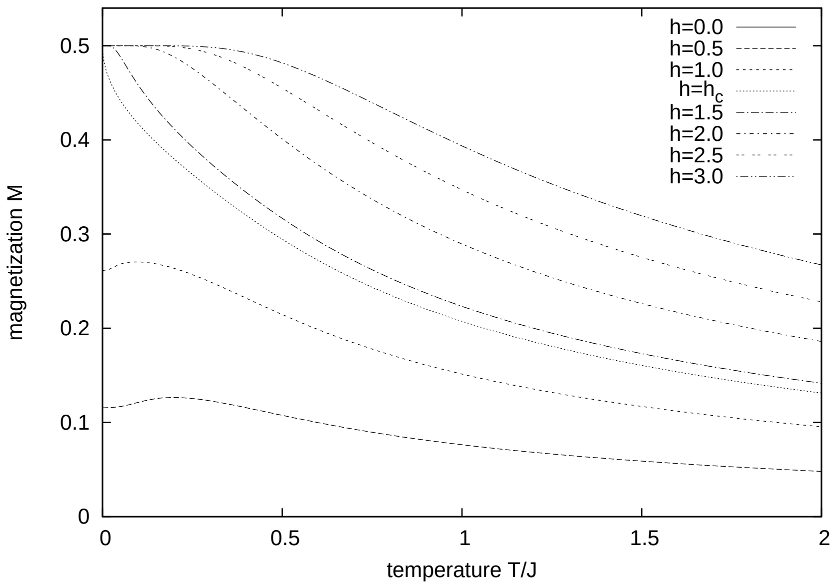


(a) Entropy vs. temperature.

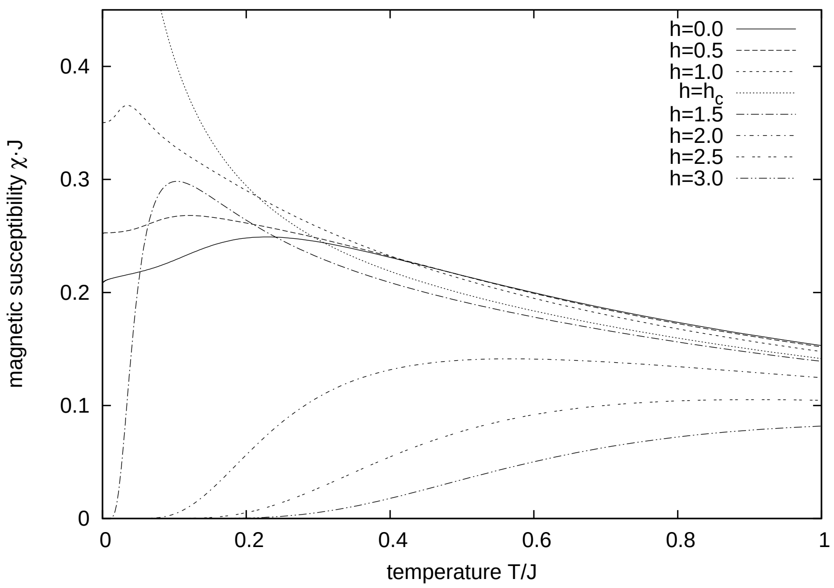


(b) Specific heat vs. temperature.

Figure 6.4: Entropy and specific heat of the $SU(4)$ spin-orbital model at $g_S = 1$, $g_\tau = 0$ for various magnetic fields. The insets show the low-temperature parts. The critical field is $h_c = 2J \ln 2 \approx 1.39J$.



(a) Magnetization vs. temperature.



(b) Magnetic susceptibility vs. temperature.

Figure 6.5: Magnetization and magnetic susceptibility of the $SU(4)$ spin-orbital model at $g_S = 1$, $g_\tau = 0$ for various magnetic fields. The critical field is $h_c = 2J \ln 2 \approx 1.39J$.

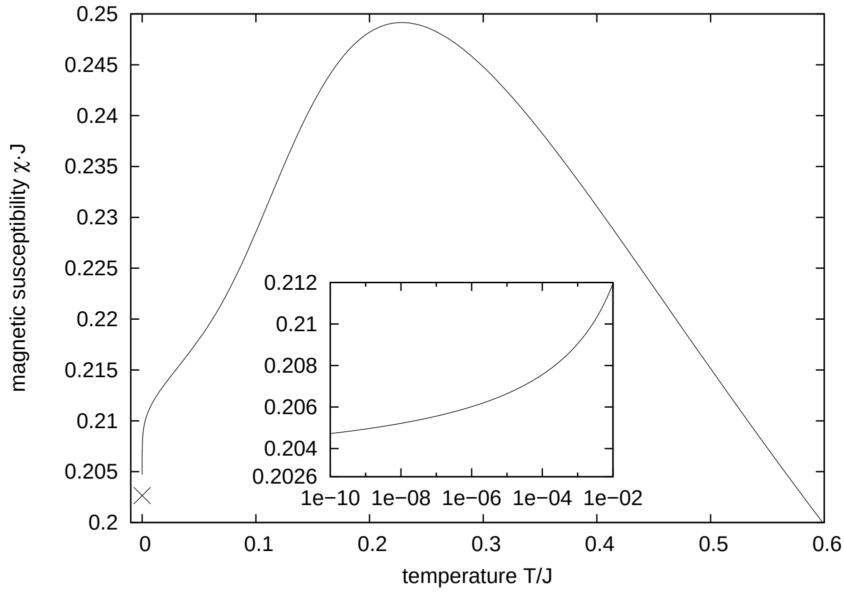


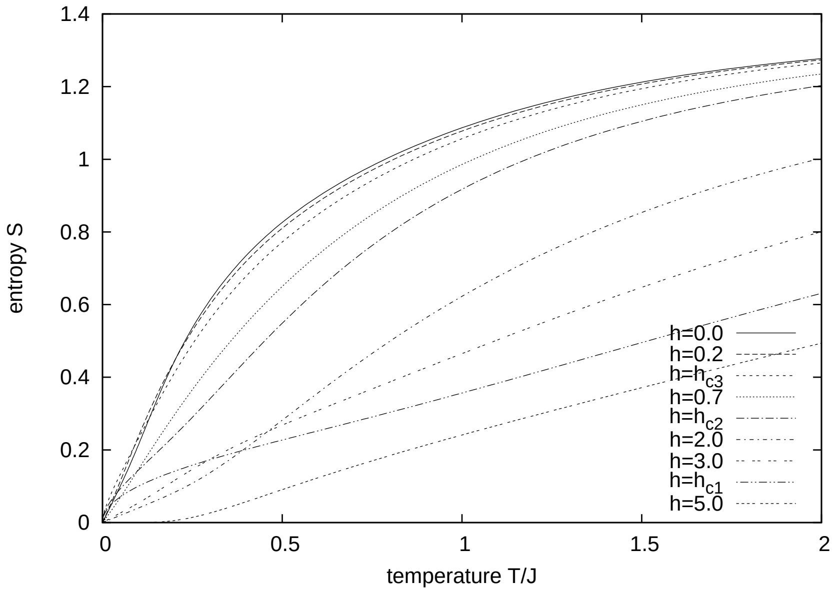
Figure 6.6: Magnetic susceptibility of the $SU(4)$ spin-orbital model at $h = 0$ for temperatures down to $T/J = 10^{-10}$. The cross denotes the ground-state value $\chi(0) \cdot J = 2/\pi^2 \approx 0.2026$. The inset shows the low-temperature part of the susceptibility using a logarithmic scale.

Next, we consider the case $g_S = 1$ and $g_\tau = 2$, where the magnetic field also couples to the pseudospin. The general chemical potentials are then given by

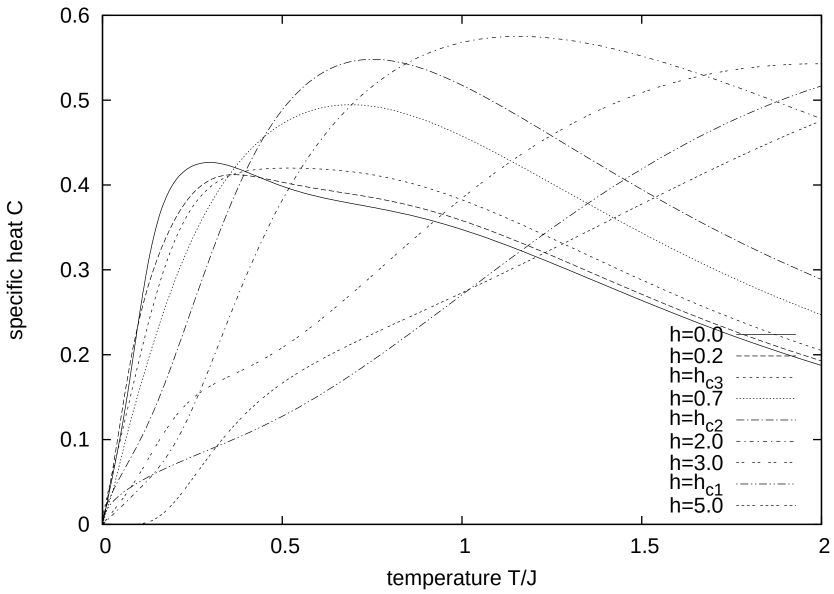
$$\mu_1 = 3h/2\,, \qquad \mu_2 = h/2\,, \qquad \mu_3 = -h/2\,, \qquad \mu_4 - 3h/2\,. \tag{6.9}$$

Note that this case also corresponds to a spin-3/2 interpretation of the model, confer Section 2.2.2. Now all three possible types of phase transitions are present. Numerical results for this case showing the rich resulting structure are plotted in Figures 6.7 and 6.8. Again, the low-temperature susceptibility at $h = 0$ shows the characteristic singular behaviour. The highest of the three critical magnetic fields is exactly $h_{c1} = 4$, while the other two have to be calculated numerically as only the lower bounds $h_{c2} > 4J\ln(2)/3 \approx 0.924J$ and $h_{c3} > J(\pi/(2\sqrt{3}) - \ln(3)/2) \approx 0.358J$ are known explicitly. We find the remaining critical fields to be $h_{c2} \approx 0.941J$ and $h_{c3} \approx 0.370J$. Note the numbering of the critical fields, where at each field h_{cj} the number of involved degrees of freedom changes from j to $j + 1$. The advantage of this naming convention is that the critical fields h_{c1} and h_{c2} also appear in the spin-1 interpretation of the $sl(3)$-symmetric Uimin-Sutherland model, while only h_{c1} remains in the spin-1/2 Heisenberg model. Likewise, all three critical fields are present in the spin-$(n-1)/2$ interpretation of the general $sl(n)$-symmetric Uimin-Sutherland model.

We are also able to calculate the complete ground-state phase diagram of the spin-orbital model. Figure 6.9 on page 95 shows the result in dependence on the magnetic field h and the orbital Landé factor g_τ, while the Landé factor for the spins is fixed at $g_S = 1$. We can identify five different phases. If the magnetic field is above h_{c1} (I), all spins and orbitals are

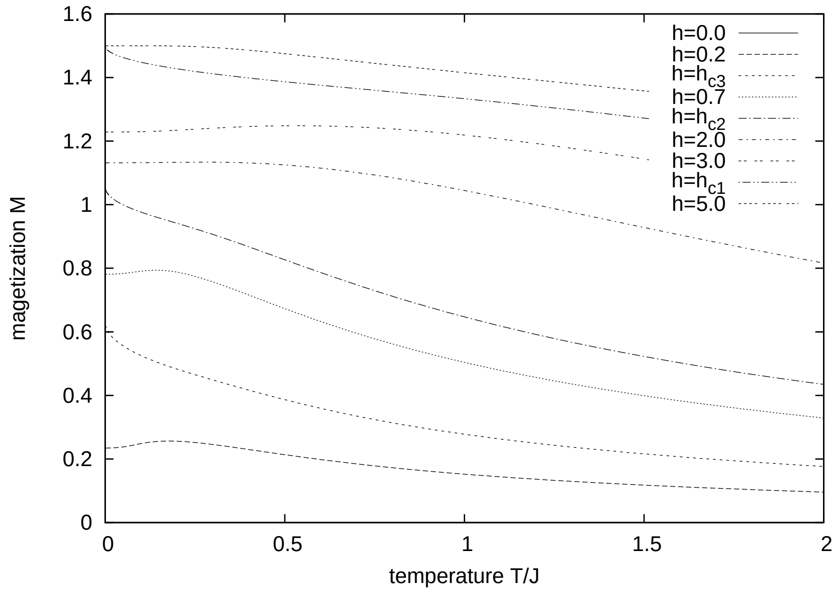


(a) Entropy vs. temperature.

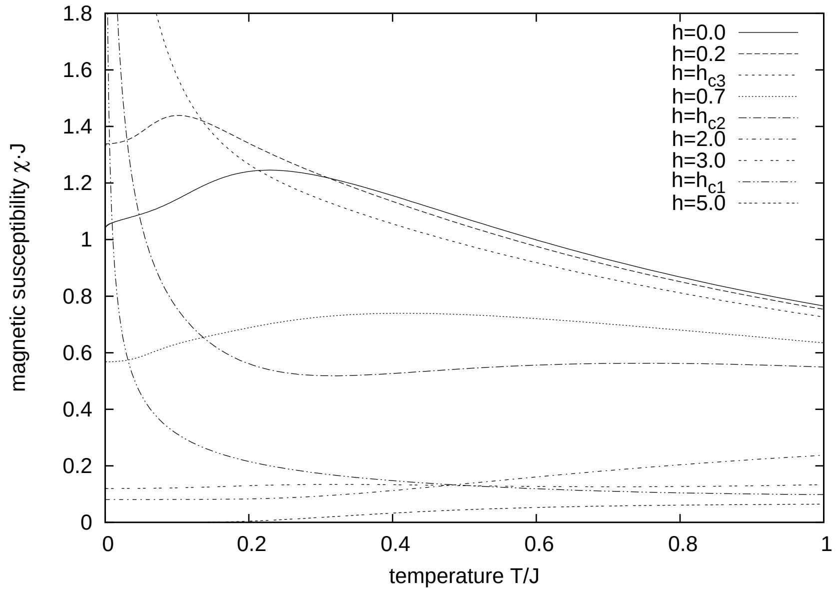


(b) Specific heat vs. temperature.

Figure 6.7: Entropy and specific heat of the $SU(4)$ spin-orbital model at $g_S = 1$, $g_\tau = 2$ for various magnetic fields. The critical fields are $h_{c1} = 4J$, $h_{c2} \approx 0.941J$ and $h_{c3} \approx 0.370J$. Note the numbering of the critical fields as discussed on the preceding page.



(a) Magnetization vs. temperature.



(b) Magnetic susceptibility vs. temperature.

Figure 6.8: Magnetization and magnetic susceptibility of the $SU(4)$ spin-orbital model at $g_S = 1$, $g_\tau = 2$ for various magnetic fields. The critical fields are $h_{c1} = 4J$, $h_{c2} \approx 0.941J$ and $h_{c3} \approx 0.370J$. Note the numbering of the critical fields as discussed on page 92.

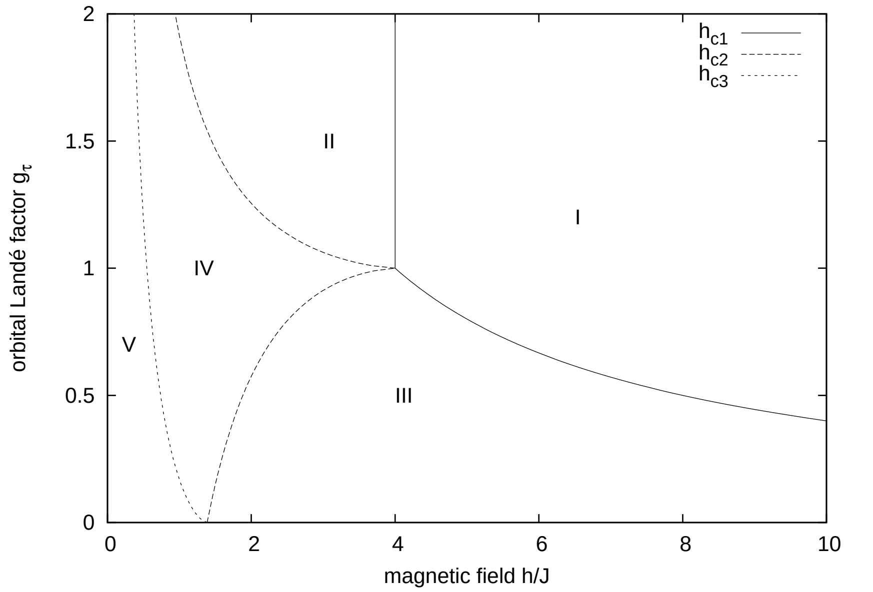


Figure 6.9: Ground-state phase diagram of the $SU(4)$ spin-orbital model depending on the magnetic field h and the Landé factor g_τ, where $g_S = 1$ is held constant. There exist five different phases (I–V), for details see the text on page 92.

fully polarized. Between h_{c1} and h_{c2}, there are two distinct regions with $g_\tau > 1$ (II) and $g_\tau < 1$ (III), respectively. In the former region, the orbitals are fully polarized, while the spins are only partially aligned. In the latter case, it is the other way around. For exactly $g_\tau = 1$, we have a direct transition from phase I to phase IV, because $h_{c1} = h_{c2}$. For a magnetic field below h_{c2}, but above h_{c3} (IV), both spins and orbitals are partially polarized, while the state $|{\downarrow_S}{\downarrow_\tau}\rangle$ is still completely suppressed. For $h < h_{c3}$ (V) all possible spin configurations contribute to the ground state. Note that h_{c1} tends to infinity for $g_\tau \to 0$. As we have seen before, only one phase transition survives for $g_\tau = 0$, where the magnetic field couples only to the spins. Note also that the phase diagram presented here is qualitatively in perfect agreement with the one presented in [32], where a finite system of 200 sites has been investigated.

6.3.3 The two-leg spin-1/2 ladder model

The thermodynamics of the two-leg spin-1/2 ladder model introduced in Section 2.2.5 has been previously investigated in [7, 8] based on high-temperature expansions of Tsuboi's NLIEs [90]. The results have moreover been compared to experimental realizations. However, in all cases that have been investigated in these papers, the rung coupling was chosen to be large in comparison to the intrachain coupling, $J_\perp \gg J_\parallel$, leading to a gapped ground state where only the rung singlet state survives. Note that under these circumstances, the high-temperature expansion gives reasonable results even in the low-temperature regime.

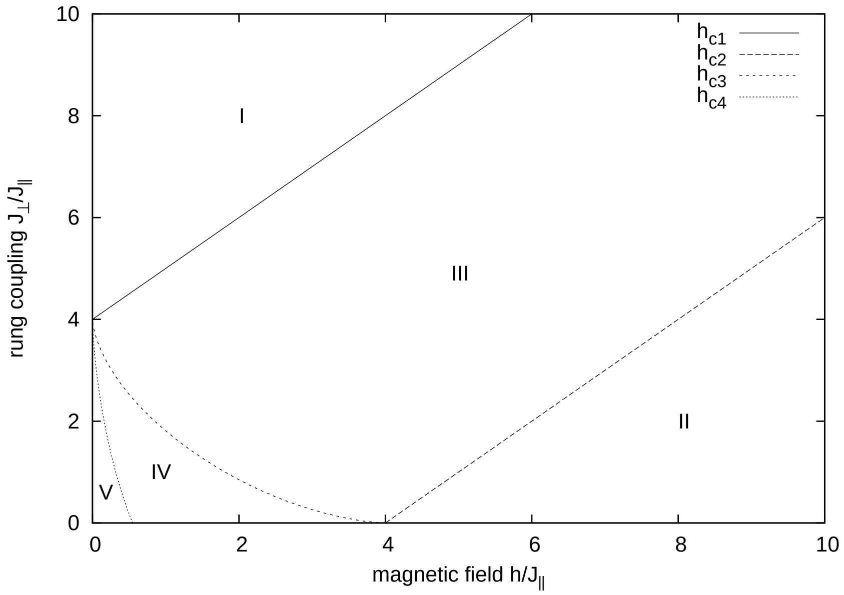
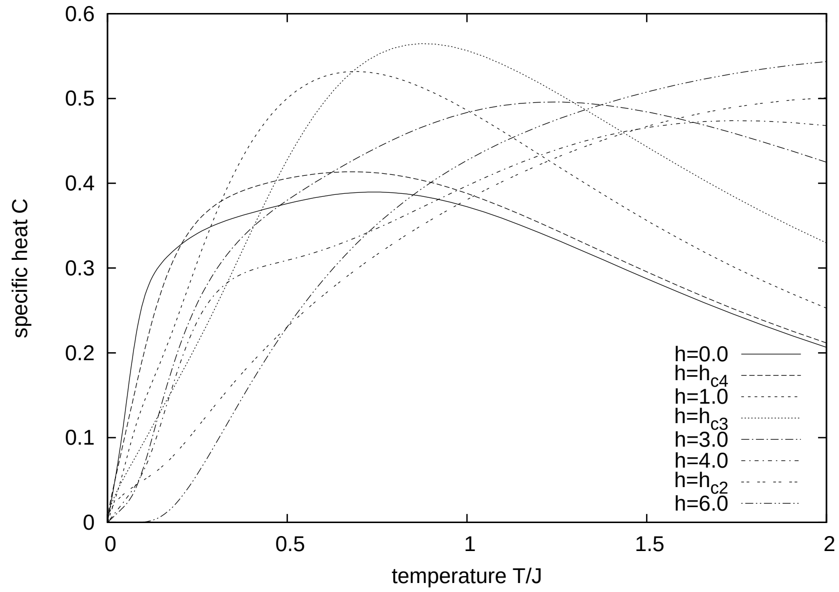


Figure 6.10: Ground-state phase diagram of the two-leg spin-1/2 ladder model depending on the ratio of the coupling constants $J_\perp/J_\parallel$ and the magnetic field h. There exist five different phases (I–V), for details see the text on this page.

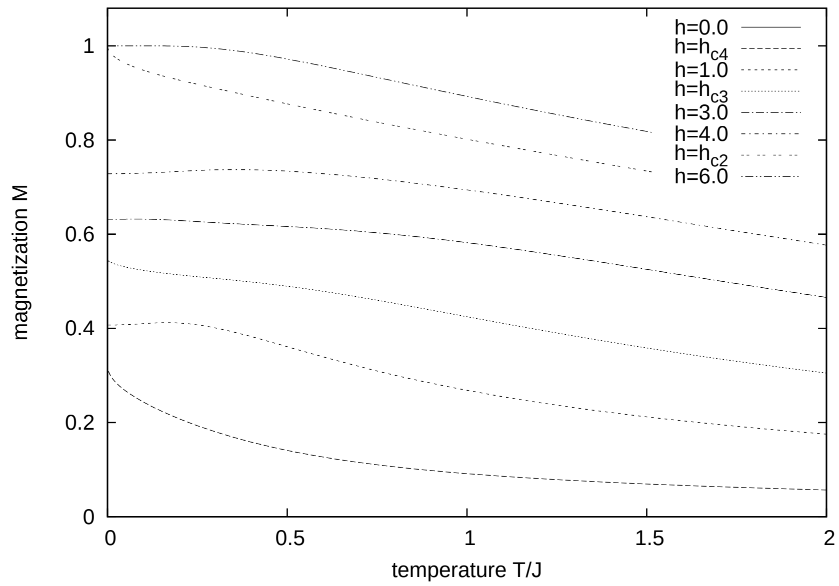
On the basis of our NLIEs, we are able to calculate accurate low-temperature results for an arbitrary choice of the coupling constants. Here, the ground state may as well be a mixture of the rung singlet and all rung triplet states.

The ground-state phase diagram of the model depending on the ratio of the coupling constants $J_\perp/J_\parallel$ and the magnetic field h is shown in Figure 6.10. Note that the first two critical fields are exactly given by $h_{c1} = J_\perp - 4J_\parallel$ and $h_{c2} = J_\perp + 4J_\parallel$, while h_{c3} and h_{c4} have to be calculated numerically. There exist five different phases. As noted above, only the rung singlet state is present in the ground state if the ratio $J_\perp/J_\parallel$ is greater than four and the magnetic field is below h_{c1} (I). Likewise, the ground state is fully polarized if the magnetic field is above h_{c2} (II). In the intermediate region (III), both the rung singlet state and the fully polarized triplet state are present, while the other two basis states are still completely suppressed. If the ratio $J_\perp/J_\parallel$ is below four and the magnetic field is between h_{c3} and h_{c4} (IV), also the neutral triplet state enters the ground state. Only if the magnetic field is smaller than h_{c4} (V), all four basis states contribute to the ground state.

In order to show the behaviour in the different regimes, we have calculated results for both $J_\perp/J_\parallel = 1$ and $J_\perp/J_\parallel = 5$ and various magnetic fields. The data for $J_\perp/J_\parallel = 1$ are shown in Figures 6.11 and 6.12. In this case, the critical fields are $h_{c2} = 5J_\parallel$, $h_{c3} \approx 1.81J_\parallel$ and $h_{c4} \approx 0.318J_\parallel$. All three phase transitions are clearly visible from the low-temperature magnetic susceptibility, which diverges at the critical fields. Above h_{c2}, where the ground state is fully polarized, the low-temperature susceptibility and the zero-temperature slope of the specific heat eventually vanish.



(a) Specific heat vs. temperature.



(b) Magnetization vs. temperature.

Figure 6.11: Specific heat and magnetization of the two-leg spin-1/2 ladder model at $J_\perp/J_\| = 1$ for various magnetic fields. The critical fields are $h_{c2} = 5J_\|$, $h_{c3} \approx 1.81J_\|$ and $h_{c4} \approx 0.318J_\|$.

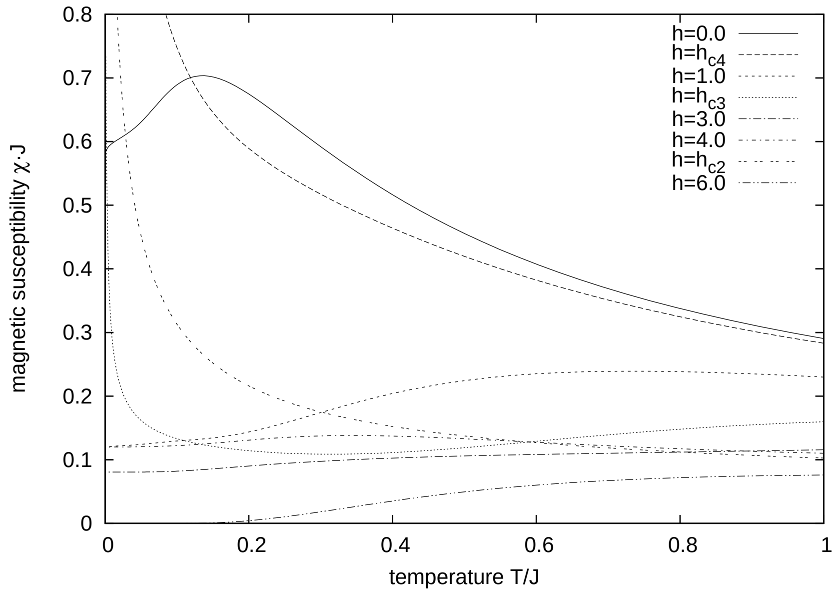


Figure 6.12: Magnetic susceptibility of the two-leg spin-1/2 ladder model at $J_\perp/J_\parallel = 1$ for various magnetic fields. The critical fields are $h_{c2} = 5J_\parallel$, $h_{c3} \approx 1.81J_\parallel$ and $h_{c4} \approx 0.318J_\parallel$.

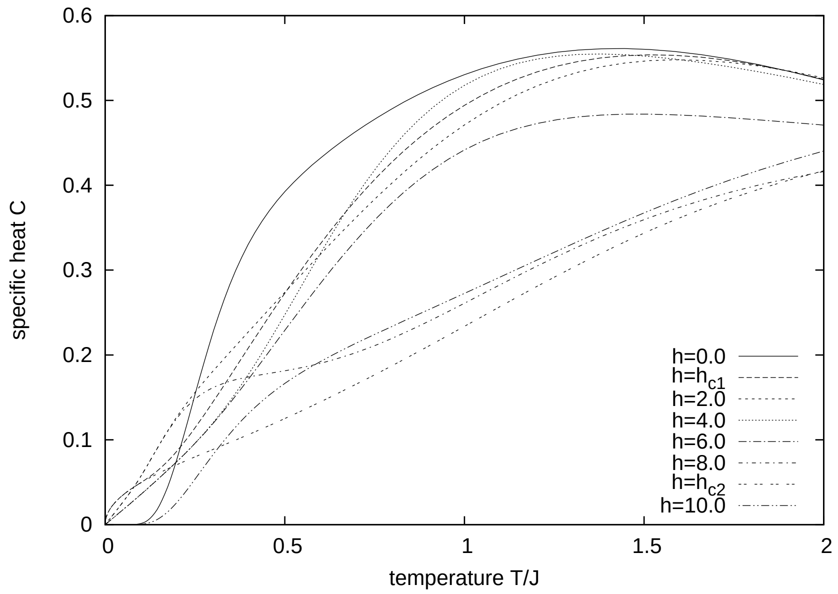
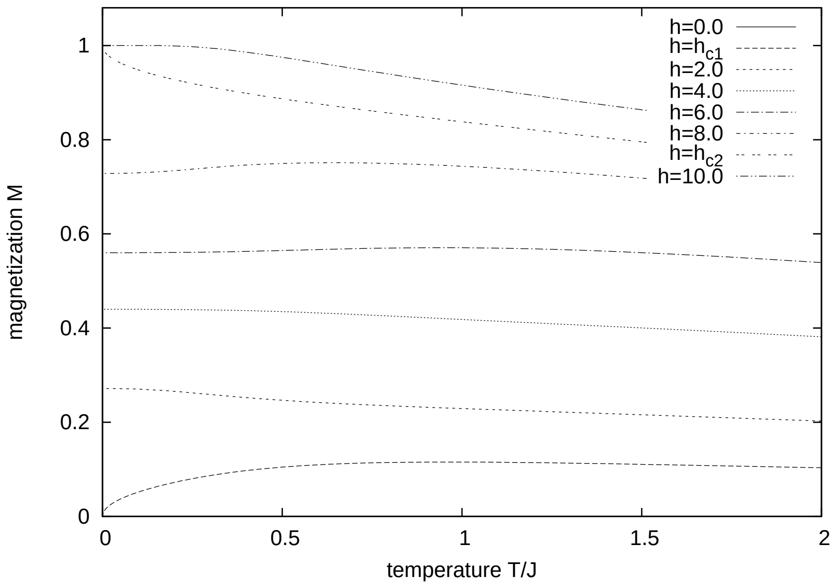
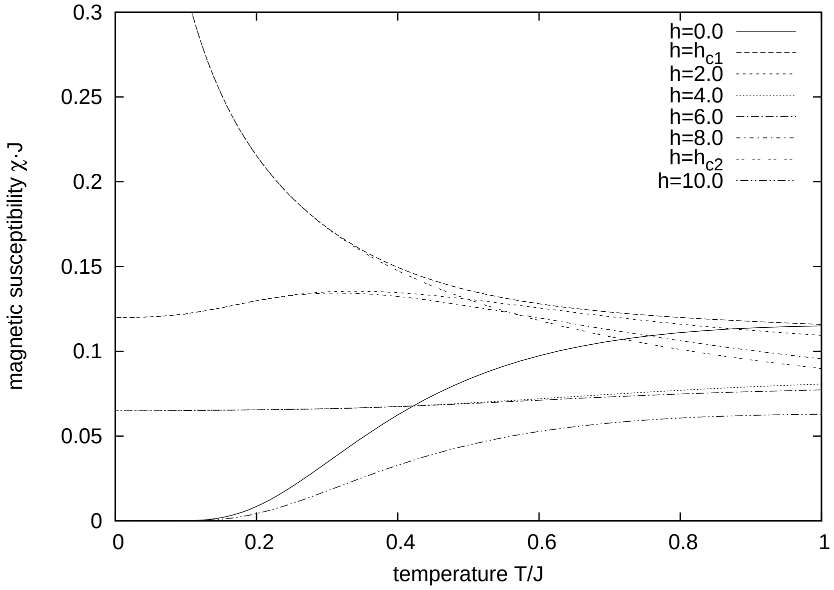


Figure 6.13: Specific heat of the two-leg spin-1/2 ladder model at $J_\perp/J_\parallel = 5$ for various magnetic fields. The critical fields are $h_{c1} = J_\parallel$ and $h_{c2} = 9J_\parallel$.



(a) Magnetization vs. temperature.



(b) Magnetic susceptibility vs. temperature.

Figure 6.14: Magnetization and magnetic susceptibility of the two-leg spin-1/2 ladder model at $J_\perp/J_\parallel = 5$ for various magnetic fields. The critical fields are $h_{c1} = J_\parallel$ and $h_{c2} = 9J_\parallel$.

Figures 6.13 and 6.14 show the corresponding data for $J_\perp/J_\parallel = 5$. Here, we have the two critical fields $h_{c1} = J_\parallel$ and $h_{c2} = 9J_\parallel$. The low-temperature slope of the specific heat is found to be zero both for $h < h_{c1}$ and $h > h_{c2}$, which is expected because all degrees of freedom are frozen out in the ground state of these cases. Accordingly, also the low-temperature magnetic susceptibility vanishes there. The susceptibility diverges at the critical fields and has constant asymptotics for $h_{c1} < h < h_{c2}$. Note also the mutual similarity of the susceptibility in the vicinity of both phase transitions. The reason is that only two basis states, the rung singlet and the fully polarized triplet state, basically contribute to the low-temperature behaviour here and that the nature of the transitions is the same at both critical fields.

6.3.4 The Essler-Korepin-Schoutens model

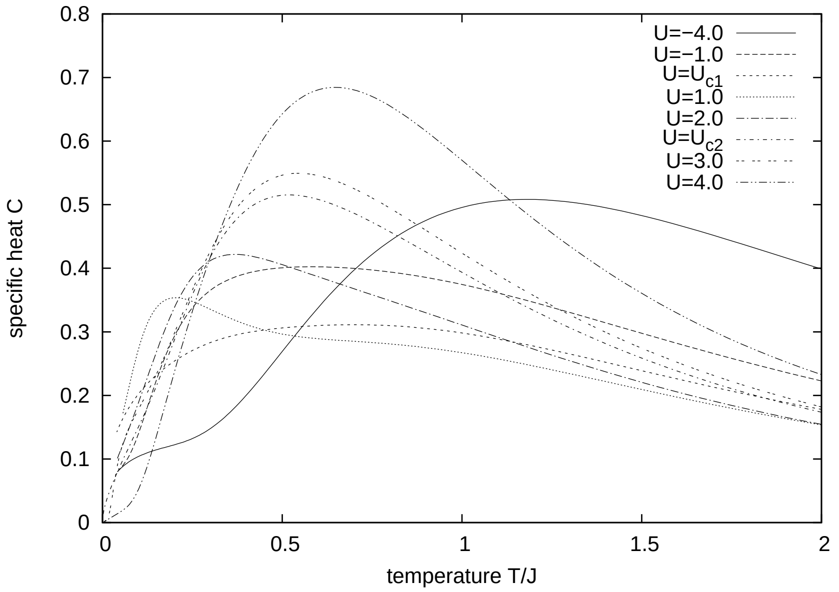
Next, we consider the Essler-Korepin-Schoutens model introduced in Section 2.2.6. Recall that the interaction is controlled by several parameters. Besides the coupling constant J, we can adjust the Hubbard parameter U, which determines whether the model favours single or double occupation of sites, the chemical potential μ and the external magnetic field h. However, instead of the chemical potential μ, we would rather like to directly control the density of states n. Since we are easily able to derive n and the compressibility κ for a fixed value of μ using our NLIEs, we can use Newton's method to iteratively adjust μ to achieve some given value for n. Therefore, we iteratively use

$$\mu' = \mu - \frac{n(\mu) - d}{\kappa(\mu)}\,, \tag{6.10}$$

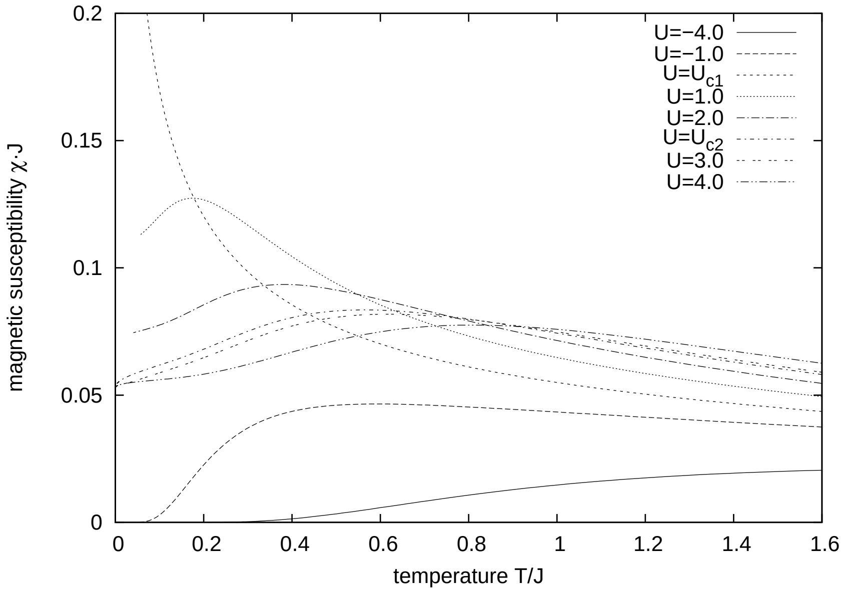
where μ' is the updated chemical potential and d the target density. However, this approach only works as long as the slope of $n(\mu)$ is neither too flat nor too steep. Note that this imposes a limitation for the evaluation at low temperatures.

The ground-state phase diagram of the model has been determined in the original work of Essler, Korepin and Schoutens [25]. Nevertheless, no numerical results have previously been calculated for finite temperature. Let us first turn to the case of half filling, where $n(\mu) = 1$, and without an external magnetic field. We know from the phase diagram that there exist two critical values for the Hubbard parameter, $U_{c1} = 0$ and $U_{c2} = 4J \ln 2 \approx 2.77J$. Their meaning is the following: If U is negative, only doubly occupied or empty sites are present in the ground state. For $U > U_{c1}$ the electron pairs start to dissolve, until above U_{c2} only single electrons are left at each site.

Figures 6.15 and 6.16 show the numerical data for the specific heat, magnetic susceptibility, compressibility and the proportion of single occupancy derived by our NLIEs. Note that the density $n = 1$ corresponds to a constant chemical potential $\mu = 0$, so no additional calculation of μ is necessary here. The phase transitions at both critical Hubbard parameters are clearly exposed by the data. For instance, we see that the low-temperature magnetic susceptibility drops to zero below U_{c1}, which is expected since no single spins are left in the ground state of this phase. On the other hand, the susceptibility diverges exactly at U_{c1} and has constant asymptotics for all values of U above this critical value. For $U \geq U_{c2}$, the ground-state value does no longer change, because only single electrons are left, and the

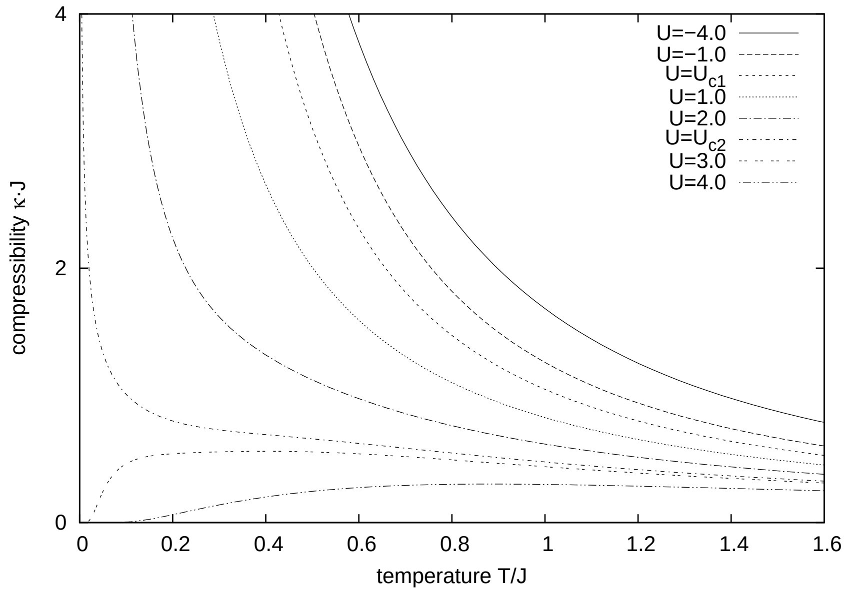


(a) Specific heat vs. temperature.

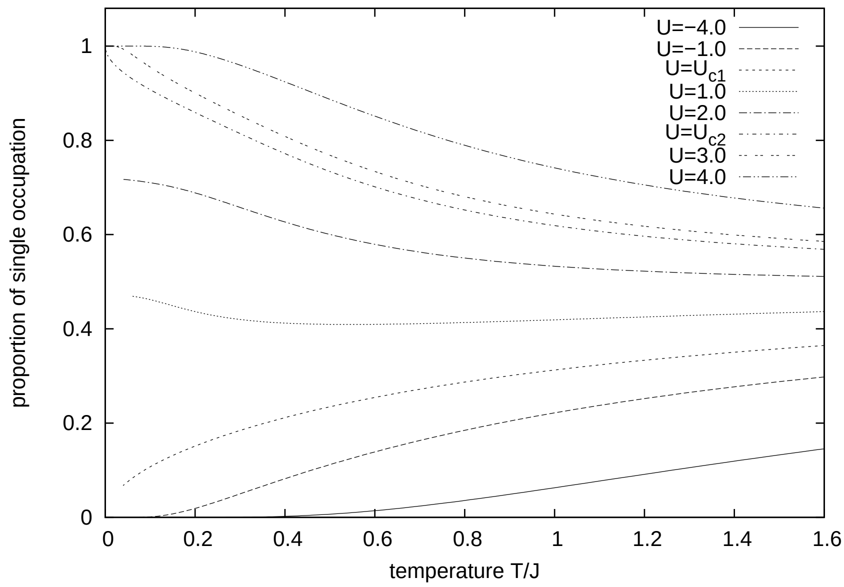


(b) Magnetic susceptibility vs. temperature.

Figure 6.15: Specific heat and magnetic susceptibility of the Essler-Korepin-Schoutens model at half filling for various Hubbard parameters U. The critical parameters are $U_{c1} = 0$ and $U_{c2} = 4J \ln 2 \approx 2.77J$.

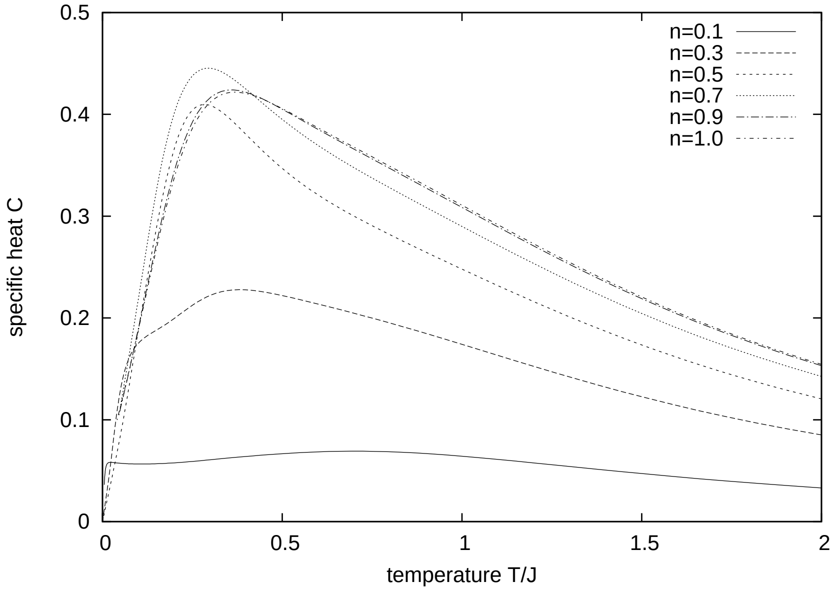


(a) Compressibility vs. temperature.

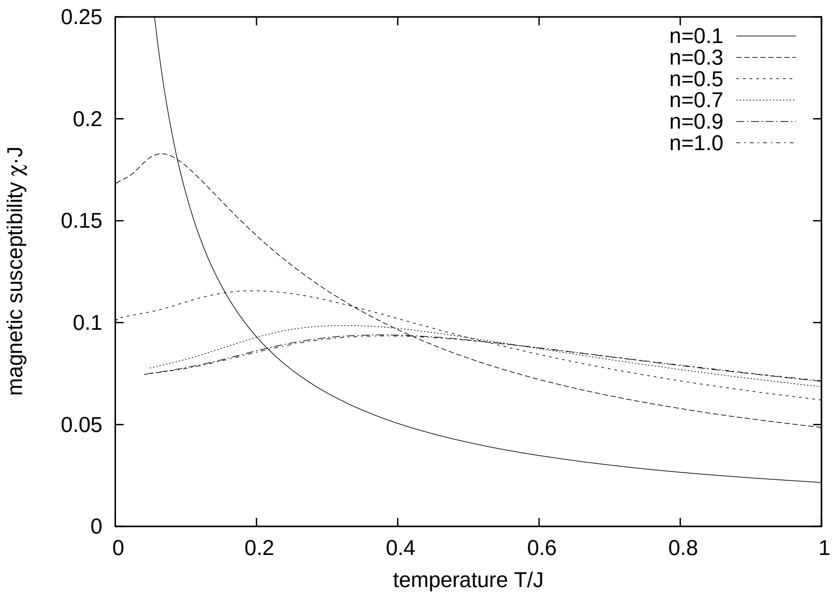


(b) Proportion of single occupation vs. temperature.

Figure 6.16: Compressibility and proportion of sites occupied by single electrons of the Essler-Korepin-Schoutens model at half filling for various Hubbard parameters U. The critical parameters are $U_{c1} = 0$ and $U_{c2} = 4J \ln 2 \approx 2.77J$.

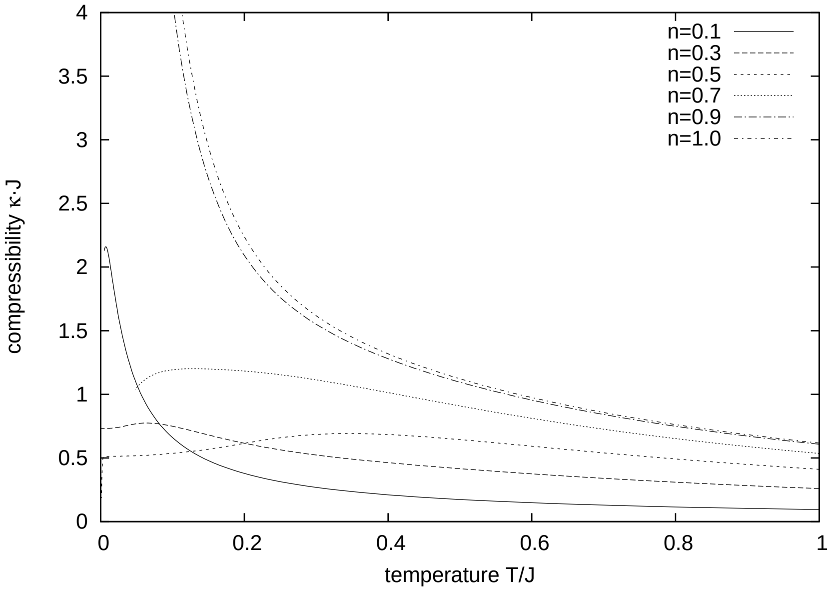


(a) Specific heat vs. temperature.

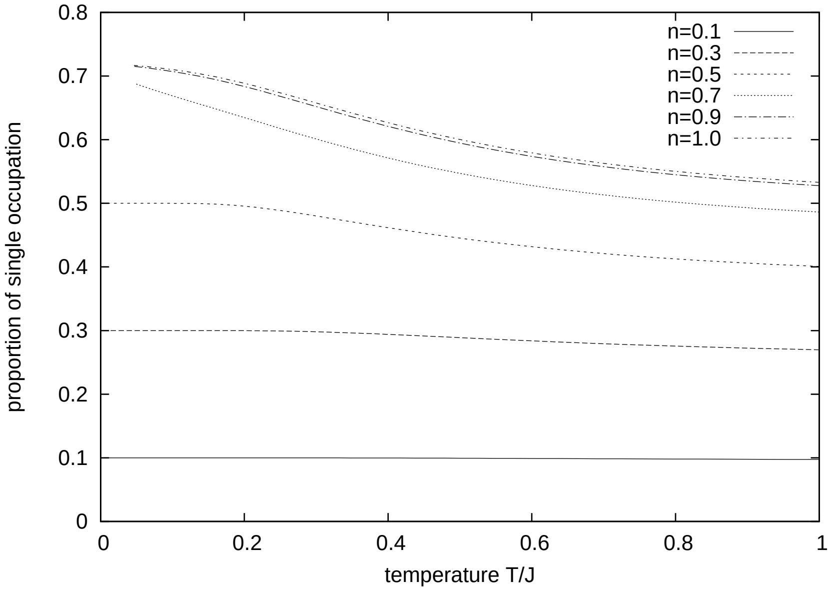


(b) Magnetic susceptibility vs. temperature.

Figure 6.17: Specific heat and magnetic susceptibility of the Essler-Korepin-Schoutens model at Hubbard parameter $U = 2.0J$ for various densities.

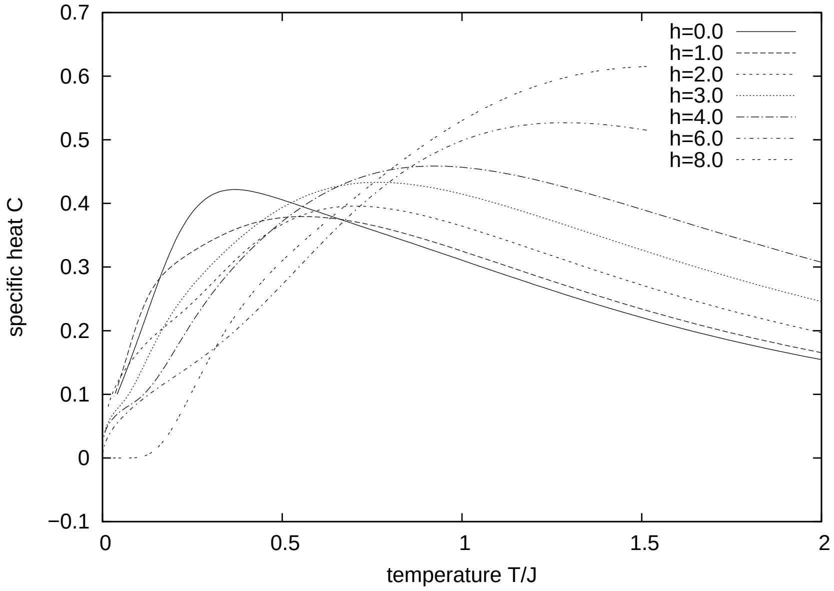


(a) Compressibility vs. temperature.

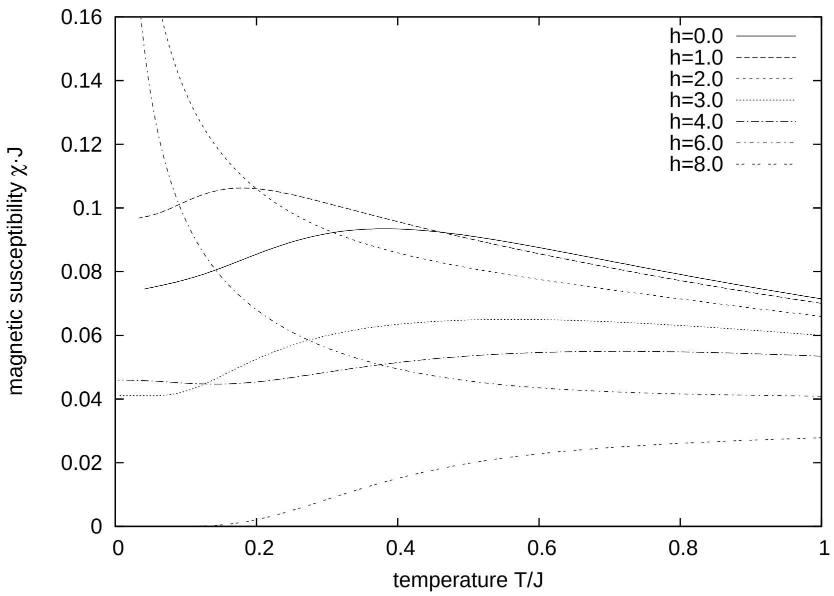


(b) Proportion of single occupation vs. temperature.

Figure 6.18: Compressibility and proportion of sites occupied by single electrons of the Essler-Korepin-Schoutens model at Hubbard parameter $U = 2.0J$ for various densities.

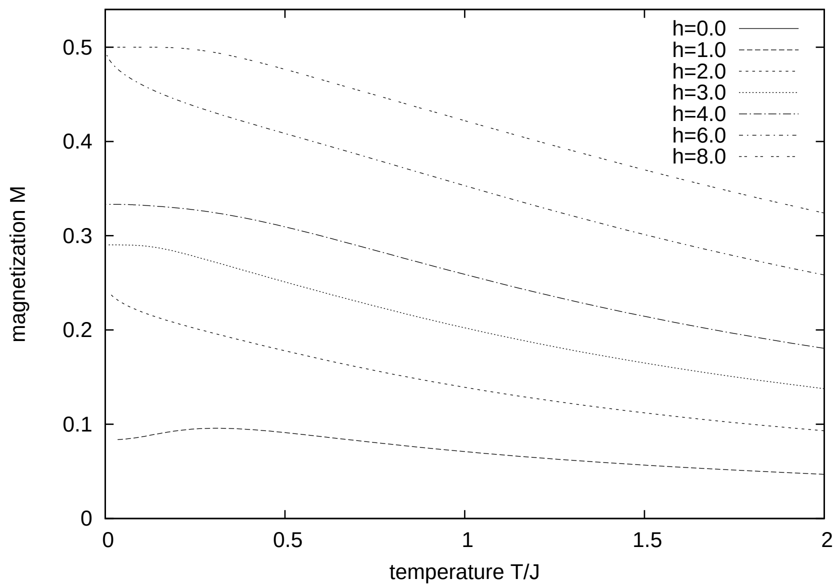


(a) Specific heat vs. temperature

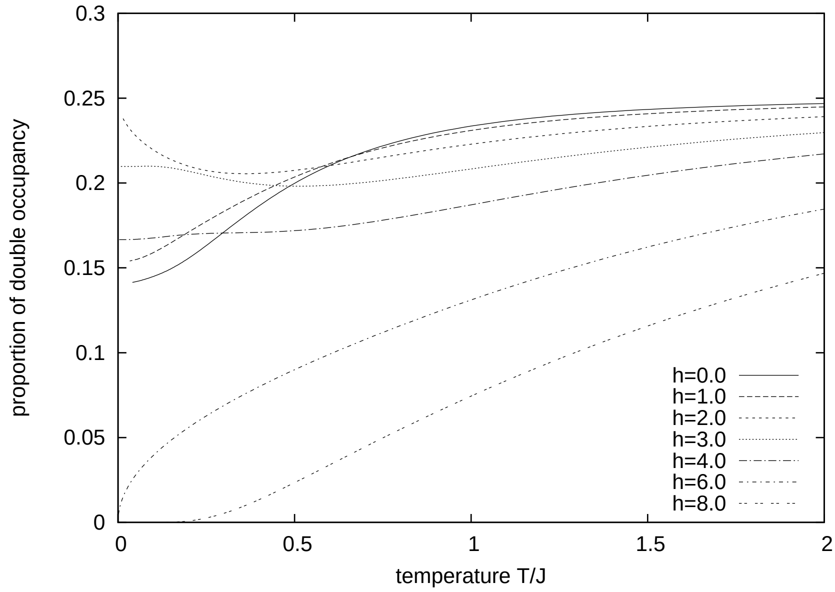


(b) Magnetic susceptibility vs. temperature

Figure 6.19: Specific heat and magnetic susceptibility of the Essler-Korepin-Schoutens model at half filling and Hubbard parameter $U = 2.0J$ for various magnetic fields.



(a) Magnetization vs. temperature



(b) Proportion of double occupancy vs. temperature

Figure 6.20: Magnetization and proportion of double occupancy of the Essler-Korepin-Schoutens model at half filling and Hubbard parameter $U = 2.0J$ for various magnetic fields.

ground state is equal to that of the usual spin-1/2 Heisenberg chain. The compressibility diverges for all values below U_{c2} and suddenly drops to zero above. This is also expected, because the half-filled chain with singly occupied sites admits no density other than $n = 1$. The data for the specific heat are less enlightening. We see that the zero-temperature slope changes from infinity to zero at U_{c2}, but since the slope of $n(\mu)$ diverges around $\mu = 0$ for $T \to 0$, we are unable to get reliable results for small temperatures if the Hubbard parameter is below U_{c2}.

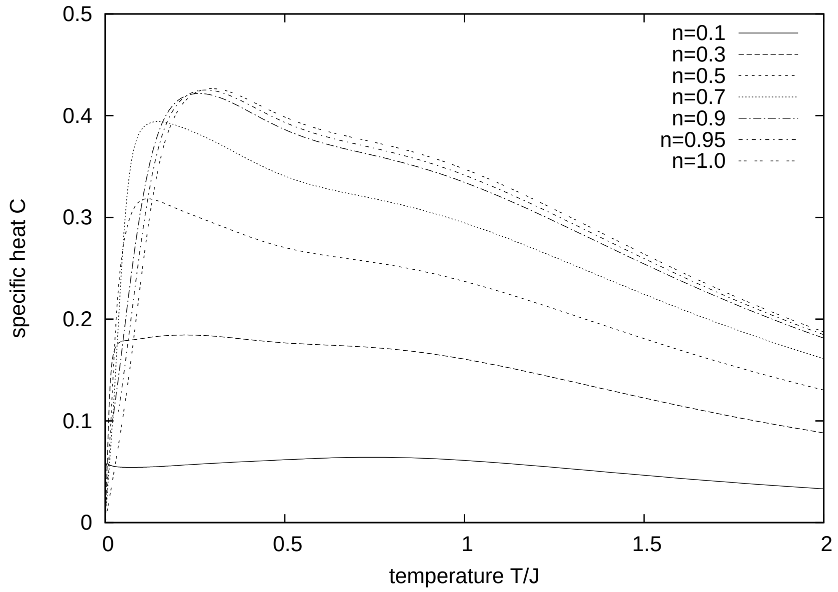
We have also calculated data for various other densities, where we have fixed the Hubbard parameter at some intermediate value $U = 2.0J$. The results are shown in Figures 6.17 and 6.18. Note that we have only considered densities between zero and half filling, since the results are symmetric to those for densities between half and complete filling, where just the role of empty and doubly occupied basis states is interchanged. We see that for densities up to approximately $n = 0.7$ only singly occupied sites are present in the ground state. Above this value, however, the positive Hubbard constant is no longer large enough to suppress double occupation. As a consequence of this phase transition, the low-temperature compressibility diverges and the magnetic susceptibility decreases.

Finally, Figures 6.19 and 6.20 show data for the specific heat and magnetic susceptibility at half filling, fixed Hubbard constant $U = 2.0J$ and an external magnetic field. Naturally, the magnetization increases with the magnetic field as the single spins start to align with the field, until all single spins are polarized above $h_{c1} \approx 2.0J$. The corresponding phase transition is most clearly indicated by the diverging zero-temperature susceptibility at the critical value. Simultaneously, we notice that the number of doubly occupied sites increases slightly. The reason is that electron pairs are neutral with respect to the magnetic field. Above h_{c1}, however, also the electron pairs start to break up, until all sites are finally filled with single, fully polarized electrons above the second critical magnetic field $h_{c2} \approx 6.0J$. Again, the phase transition is indicated by a diverging ground-state susceptibility at the critical value. Above this value, the specific heat and the susceptibility of the ground state drop to zero, since now all degrees of freedom are frozen out.

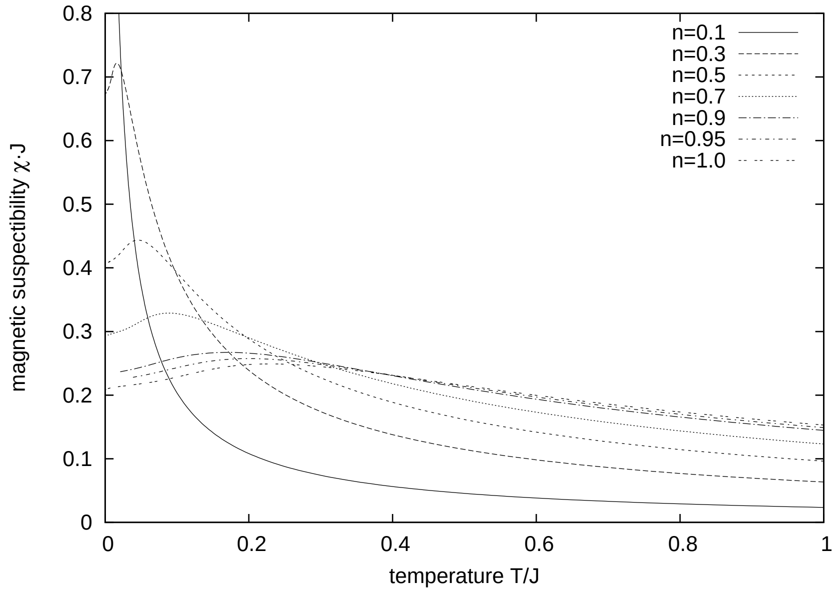
6.3.5 The $SU(4|1)$ spin-orbital model with mobile defects

As stated before, the $SU(4|1)$ spin-orbital model is an extension of the $SU(4)$ spin-orbital model, where mobile defects are additionally allowed and the hopping term is the same as in the t-J model. Like in the previous section, the particle density n can be controlled only indirectly by iteratively determining the corresponding chemical potential $\mu(n)$.

Figures 6.21 and 6.22 show the thermodynamical properties of the model at various particle densities, where the magnetic field is chosen to only couple to the spin degrees of freedom. It can clearly be seen that for increasing particle density n, as expected, the specific heat and magnetic susceptibility tend to those of the $SU(4)$ spin-orbital model. The compressibility meanwhile drops to zero, since the number of holes is reduced. For low temperatures and particle densities close to one, the density unfortunately becomes independent of the chemical potential. Therefore, we have no reliable results in the low-temperature area in these cases.



(a) Specific heat vs. temperature.



(b) Magnetic susceptibility vs. temperature.

Figure 6.21: Specific heat and magnetic susceptibility of the $SU(4|1)$ spin-orbital model at $g_S = 1$, $g_\tau = 0$ for various particle densities. The data for $n = 1.0$ stems from the $SU(4)$ spin-orbital model.

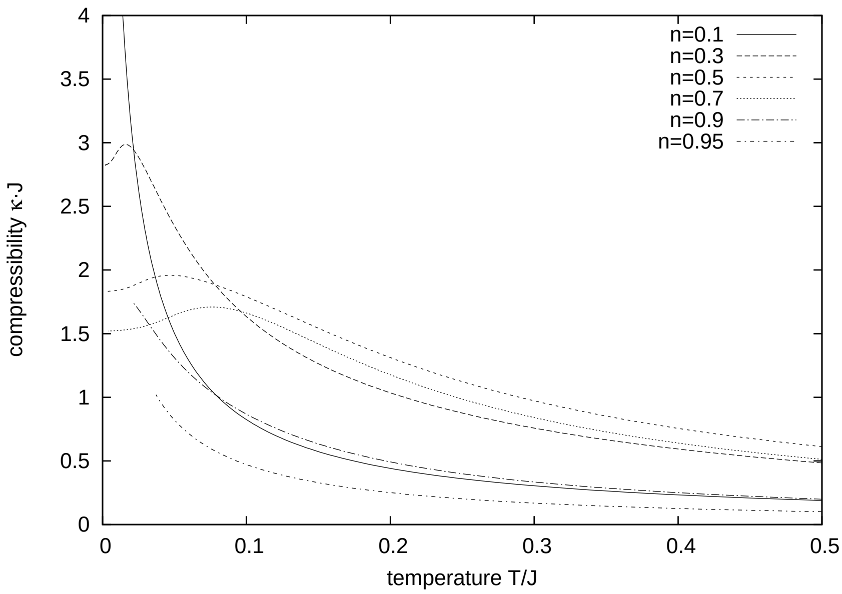


Figure 6.22: Compressibility of the $SU(4|1)$ spin-orbital model at $g_S = 1$, $g_\tau = 0$ for various particle densities.

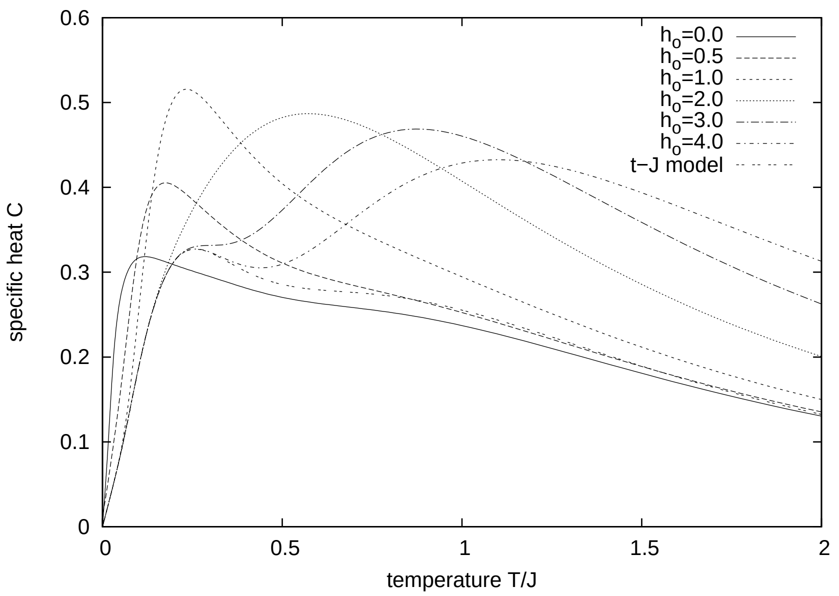
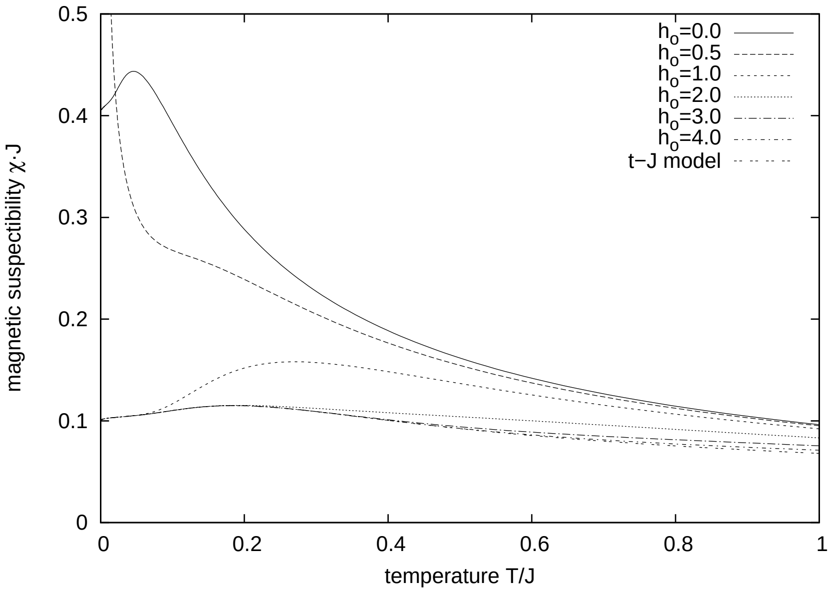
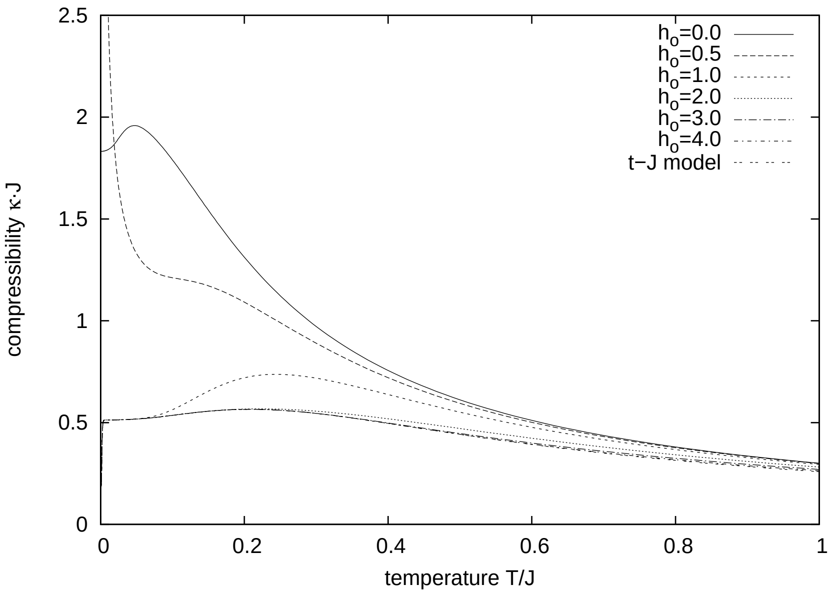


Figure 6.23: Specific heat of the $SU(4|1)$ spin-orbital model at $g_S = 1$, $g_\tau = 0$ for particle density $n = 0.5$ and various orbital fields. Results for the supersymmetric t-J model are also given.



(a) Magnetic susceptibility vs. temperature.



(b) Compressibility vs. temperature.

Figure 6.24: Magnetic Susceptibility and compressibility of the $SU(4|1)$ spin-orbital model at $g_S = 1$, $g_\tau = 0$ for particle density $n = 0.5$ and various orbital fields. Results for the supersymmetric t-J model are also given.

We have also calculated results for the limit, where the orbital degrees of freedom are gradually frozen out. This is achieved by introducing a secondary magnetic field that only affects the orientation of the pseudospins. Eventually, only three of the five basis states remain. We therefore expect the limiting case to be equivalent to the supersymmetric t-J model. The corresponding data are shown in Figures 6.23 and 6.24, where we have also plotted results for the supersymmetric t-J model for comparison. The expected behaviour can clearly be seen. Starting from the unperturbed model, the orbital degrees are gradually frozen out with rising orbital field h_o. The critical field is approximately at $h_o = 0.5J$, where we get a diverging low-temperature magnetic susceptibility and compressibility. Above the critical value, all pseudospins are polarized in the ground state. By further increasing the orbital field, we can finally see the expected convergence towards the data of the supersymmetric t-J model.

We like to stress that the numerical data again strongly support our conjectured NLIEs for this case.

Chapter 7

Summary and outlook

In this thesis, we have treated the thermodynamics of various integrable quantum chains of Uimin-Sutherland type on the basis of nonlinear integral equations (NLIEs). There exist several approaches to derive such NLIEs, confer [34, 90]. Nevertheless, only the approach developed in [40, 41] is known to allow for an efficient numerical solution for arbitrary finite temperature and rather general chemical potentials. The approach is based on a Trotter-Suzuki mapping of the quantum chain to an equivalent two-dimensional classical model. In case of the Uimin-Sutherland model the classical counterpart is an inhomogeneous Perk-Schultz model, where alternating R matrices appear in vertical direction. For the latter model one defines the so-called quantum transfer matrix (QTM), which is basically the inhomogeneous column-to-column transfer matrix with an additional spectral parameter. This allows for the formulation of the partition function of the Uimin-Sutherland model solely in terms of the QTM. The big advantage is that the thermodynamical limit can be performed exactly, and it follows that the thermodynamical properties of the model depend on just the largest eigenvalue of the QTM. Moreover, the QTM can be diagonalized by means of the Bethe ansatz, since the underlying R matrices are solutions to the Yang-Baxter equation and thus a whole commuting family of QTMs exists. The calculation of the Bethe ansatz roots still turns out to be problematic. Therefore, the Bethe ansatz equations are encoded into a set of suitable auxiliary functions, which turn out to be determined by a finite set of coupled NLIEs containing only convolution-type integrals. For numerical investigations, the NLIEs can then be solved efficiently by iteration, where the fast Fourier transform is used to calculate the convolutions. In contrast to the other approaches, however, no general construction of the NLIEs for arbitrary cases of the Uimin-Sutherland model is known. Previous to our work, results were known for up to three components at most [27, 36, 37].

In this work, we have succeeded in finding suitable auxiliary functions for all four-component cases of the Uimin-Sutherland model and have derived the corresponding closed sets of NLIEs. Several limiting cases have been considered in order to check the validity of our new NLIEs. In the zero-temperature limit, for example, the NLIEs linearize and are equal to those, one can get either from the zero-temperature limit of the traditional TBA equations or directly from the Bethe ansatz equations for the Hamiltonian [34]. The previously known auxiliary functions and NLIEs are recovered by freezing out one or more of the basis states with the help of the general chemical potentials. Moreover, we could show that—after a slight modification—the auxiliary functions can be used to exactly truncate the TBA equations at some arbitrary fusion level, which extends the result of [76]. Note that our treatment covers only the fundamental representation of the underlying algebra. A further generalization to higher-level representations should nevertheless be straightforward.

Unfortunately, the general construction of the auxiliary functions for arbitrary higher-rank cases of the Uimin-Sutherland model stays out of reach. All new sets of auxiliary functions have basically been found by trial and error. On the level of the final NLIEs, however, certain conjectures are possible. They are based on the observation that the number of required auxiliary functions for the $sl(n)$-symmetric case should always be equal to the sum of the dimensions of all fundamental representations of $sl(n)$, that is $2^n - 2$, and the requirement that both the zero-temperature limit and the reduction of the basis states reproduce the known results. We have shown that these assumptions are sufficient to fix all driving terms and most of the entries of the kernel matrix of the NLIEs of the $sl(n)$-symmetric case once the results for the $sl(n-1)$-symmetric case are known. Nevertheless, they provide no information on the kernel functions lying on the antidiagonal of the kernel matrix, which prevents a recursive derivation. Still, we have been able to conjecture the complete set of NLIEs for the $sl(5)$-symmetric case by exploiting some additional algebra-related structure of the kernel matrix to get the missing kernel functions. We like to stress, however, that the emergence of the additional structure is not yet understood. It is our hope that further investigations in this direction may lead to explicit results for the general $sl(n)$-symmetric case in the future. We have also been able to conjecture the general structure of the final NLIEs of the $sl(n|1)$-symmetric case, which turn out to be very similar to those of the $sl(n)$-symmetric case. Here, the number of required equations is expected to be $2^n - 1$. Based on the known results for the $sl(4)$-symmetric case, for example, we could thus fix the complete NLIEs of the $sl(4|1)$-symmetric case. Additionally, we have investigated the general structure of the exactly truncated TBA equations for both the $sl(n)$- and the $sl(n|1)$-symmetric cases.

Numerical results have been provided for all new sets of NLIEs. In order to check the validity of our results, we have compared our results for the specific heat to those obtained from the high-temperature expansions and Padé approximants derived in [91]. As expected, the data are in excellent agreement within the corresponding areas of validity. Especially for the $sl(5)$- and $sl(4|1)$-symmetric cases, for which our NLIEs have only been conjectured, this provides strong evidence for the correctness of the NLIEs and supports our assumptions on their general structure. Moreover, we have calculated thermodynamical properties for various applications of the Uimin-Sutherland model, like the $SU(4)$ spin-orbital model, the integrable two-leg spin-1/2 ladder, the Essler-Korepin-Schoutens model and the $SU(4|1)$ spin-orbital model with mobile defects, which have become accessible via our new sets of NLIEs. In contrast to previous approaches, we achieve high accuracy even at low temperatures.

However, although we have thus managed to extend the current approach to include most of the physically interesting models of higher-rank Uimin-Sutherland type, some interesting models are still not accessible in this framework. These mainly include further applications of the $sl(n)$- and $sl(n|1)$-symmetric models for $n > 5$, like the general integrable n-leg spin-1/2 ladder [9], the integrable mixed spin ladders [6] and higher-spin generalizations of the supersymmetric t-J model [58].

The treatment of models based on q-deformations of the underlying algebra is a further direction of generalization. Since the Bethe ansatz is similar for these models, this should be possible by rather straightforward modifications of the auxiliary functions. Note that in the framework of Tsuboi's NLIEs this has already been demonstrated [91, 92].

Concerning the structure of our NLIEs, another open question is the connection between the kernel matrix elements and the complete S matrix of elementary excitations [20]. In the zero-temperature case, all kernel functions can be obtained from corresponding S matrix entries [50], the generalization to finite temperature is unknown.

The systematic derivation of finite-temperature correlation lengths is yet another future task. For that, it is necessary to calculate also the next-leading eigenvalues of the QTM. The corresponding NLIEs follow from the same auxiliary functions, but modified integration contours are required due to the different distributions of the Bethe ansatz roots. Note that corresponding results already exist for the spin-1/2 Heisenberg chain [40, 41]. In the context of the TBA equations, the necessary modifications have been discussed in [38].

Future developments in the spirit of [14, 15] may also allow for the direct calculation of finite-temperature correlation functions. Note, however, that the approach is currently limited to the XXZ chain.

Appendix A

Equivalence of different grading choices

Let us consider the $sl(r|s)$-symmetric case of the Uimin-Sutherland model for some fixed numbers r and s. From the definition of the Hamiltonian (2.2) it is immediately clear that there exist $\binom{r+s}{s}$ many equivalent formulations differing only in the actual choice of grading. Changing the grading just leads to some permutation of the basis states and, accordingly, of the general chemical potentials μ_j and of the signs in the definition of the permutation operator $\pi_{j,j+1}$. In the framework of the Bethe ansatz formulation (2.57), however, the equivalence is not so obvious, since, at first glance, the expression for the eigenvalue

$$\Lambda(v) = \sum_{j=1}^{r+s} \lambda_j(v) \tag{A.1}$$

with

$$\lambda_j(v) = \phi_-(x)\phi_+(x) \frac{q_{j-1}(v - \mathrm{i}\epsilon_j)}{q_{j-1}(v)} \frac{q_j(v + \mathrm{i}\epsilon_j)}{q_j(v)} \mathrm{e}^{\beta\mu_j} \tag{A.2}$$

does not seem to be invariant under a combined permutation of gradings and general chemical potentials. The eigenvalue must be invariant, however, since the Hamiltonians are equivalent. The clue is that the polynomials $q_j(v)$ of Bethe ansatz roots, see the definition (2.59), also depend on the choice of grading. In the following, we will show how the Bethe ansatz roots have to be adapted by using the approach introduced in [31].

Suppose, we have chosen some fixed pattern of gradings $\{\epsilon\}$. Now let us investigate what happens, if we exchange just two neighbouring gradings, ϵ_j and ϵ_{j+1}, where $\epsilon_j = -\epsilon_{j+1}$. The resulting pattern shall be called $\{\widetilde{\epsilon}\}$, but let us first stick to $\{\epsilon\}$. We define the polynomial

$$p_j(v) = q_{j-1}(v - \mathrm{i}\epsilon_j) q_{j+1}(v) \mathrm{e}^{\beta\mu_j} + q_{j-1}(v) q_{j+1}(v - \mathrm{i}\epsilon_j) \mathrm{e}^{\beta\mu_{j+1}} \tag{A.3}$$

and compare it with the jth Bethe ansatz equation (2.60),

$$-1 = \frac{\lambda_j(v)}{\lambda_{j+1}(v)} = \frac{q_{j-1}(v - \mathrm{i}\epsilon_j) q_{j+1}(v)}{q_{j-1}(v) q_{j+1}(v - \mathrm{i}\epsilon_j)} \mathrm{e}^{\beta(\mu_j - \mu_{j+1})}\,, \tag{A.4}$$

which is fulfilled for all M_j many Bethe ansatz roots $v^j_{k_j}$ of the jth set and, additionally, for the $M^{(h)}_j = M_{j-1} + M_{j+1} - M_j$ many hole solutions $v'^j_{l_j}$. Obviously, the roots of the polynomial $p_j(v)$ are given by the Bethe ansatz roots and the hole solutions. We may therefore write

$$p_j(v) = q_j(v) q^{(h)}_j(v) (\mathrm{e}^{\beta\mu_j} + \mathrm{e}^{\beta\mu_{j+1}}) \,, \tag{A.5}$$

where

$$q^{(h)}_j(v) = \prod_{l_j=1}^{M^{(h)}_j} (v - v'^j_{l_j}) \tag{A.6}$$

is defined in analogy to $q(v)$. Next, we consider the quotient

$$\begin{aligned} \frac{p_j(v + \mathrm{i}\epsilon_j)}{p_j(v)} &= \frac{q_{j-1}(v) q_{j+1}(v + \mathrm{i}\epsilon_j) \mathrm{e}^{\beta\mu_j} + q_{j-1}(v + \mathrm{i}\epsilon_j) q_{j+1}(v) \mathrm{e}^{\beta\mu_{j+1}}}{q_{j-1}(v - \mathrm{i}\epsilon_j) q_{j+1}(v) \mathrm{e}^{\beta\mu_j} + q_{j-1}(v) q_{j+1}(v - \mathrm{i}\epsilon_j) \mathrm{e}^{\beta\mu_{j+1}}} \\ &= \frac{q_j(v + \mathrm{i}\epsilon_j) q^{(h)}_j(v + \mathrm{i}\epsilon_j)}{q_j(v) q^{(h)}_j(v)} \,. \end{aligned} \tag{A.7}$$

The resulting identity can be recast as

$$\begin{aligned} &\frac{q_{j-1}(v - \mathrm{i}\epsilon_j)}{q_{j-1}(v)} \frac{q_j(v + \mathrm{i}\epsilon_j)}{q_j(v)} \mathrm{e}^{\beta\mu_j} + \frac{q_j(v - \mathrm{i}\epsilon_{j+1})}{q_j(v)} \frac{q_{j+1}(v + \mathrm{i}\epsilon_{j+1})}{q_{j+1}(v)} \mathrm{e}^{\beta\mu_{j+1}} \\ &\quad = \frac{q_{j-1}(v - \mathrm{i}\epsilon_{j+1})}{q_{j-1}(v)} \frac{q^{(h)}_j(v)}{q^{(h)}_j(v - \mathrm{i}\epsilon_{j+1})} \mathrm{e}^{\beta\mu_{j+1}} + \frac{q^{(h)}_j(v)}{q^{(h)}_j(v + \mathrm{i}\epsilon_j)} \frac{q_{j+1}(v + \mathrm{i}\epsilon_j)}{q_{j+1}(v)} \mathrm{e}^{\beta\mu_j} \,. \end{aligned} \tag{A.8}$$

After multiplying both sides with $\phi_-(x)\phi_+(x)$, we find that the two terms on the left hand side are equal to our original functions $\lambda_j(v)$ and $\lambda_{j+1}(v)$, respectively. Defining

$$\widetilde{q}_j(v) = q^{(h)}_j(v + \mathrm{i}\epsilon_j) = q^{(h)}_j(v - \mathrm{i}\epsilon_{j+1}) \,, \tag{A.9}$$

we get the result

$$\begin{aligned} \lambda_j(v) + \lambda_{j+1}(v) &= \phi_-(x)\phi_+(x) \frac{q_{j-1}(v - \mathrm{i}\epsilon_{j+1})}{q_{j-1}(v)} \frac{\widetilde{q}_j(v + \mathrm{i}\epsilon_{j+1})}{\widetilde{q}_j(v)} \mathrm{e}^{\beta\mu_{j+1}} \\ &\quad + \phi_-(v)\phi_+(v) \frac{\widetilde{q}_j(v - \mathrm{i}\epsilon_j)}{\widetilde{q}_j(v)} \frac{q_{j+1}(v + \mathrm{i}\epsilon_j)}{q_{j+1}(v)} \mathrm{e}^{\beta\mu_j} \,. \end{aligned} \tag{A.10}$$

Comparing the right hand side to the left, we notice that the structure is basically the same. On the right hand side, the neighbouring gradings ϵ_j and ϵ_{j+1} and accordingly the chemical potentials have been interchanged. At the same time, the polynomial $q_j(v)$ has been replaced by $\widetilde{q}_j(v) = q^{(h)}_j(v + \mathrm{i}\epsilon_j)$. We may identify the right hand side with the expression $\lambda_j(v) + \lambda_{j+1}(v)$ obtained for the new set of gradings $\{\widetilde{\epsilon}\}$. The remaining λ functions are the same for both sets $\{\epsilon\}$ and $\{\widetilde{\epsilon}\}$.

Grading $\{\epsilon\}$		Grading $\{\widetilde{\epsilon}\}$
$\{\ldots,\epsilon_j,\epsilon_{j+1},\ldots\}$	$\mapsto$	$\{\ldots,\epsilon_{j+1},\epsilon_j,\ldots\}$
μ_j	$\mapsto$	$\widetilde{\mu}_j=\mu_{j+1}$
μ_{j+1}	$\mapsto$	$\widetilde{\mu}_{j+1}=\mu_j$
$q_j(v)$	$\mapsto$	$\widetilde{q}_j(v)=q_j^{(h)}(v+\mathrm{i}\epsilon_j)$
$q_j^{(h)}(v)$	$\mapsto$	$\widetilde{q}_j^{(h)}(v)=q_j(v+\mathrm{i}\epsilon_j)$

Table A.1: By exchanging two neighbouring gradings ϵ_j and ϵ_{j+1}, not only the general chemical potentials, but also the polynomial of Bethe ansatz roots $q_j(v)$ and the polynomial of hole solutions $q_j^{(h)}(v)$ are modified. The eigenvalue $\Lambda(v)$ therefore stays invariant.

Thus, the eigenvalue $\Lambda(v)$ stays invariant, but the set of Bethe ansatz roots and hole solutions changes. The corresponding mapping between $\{\epsilon\}$ and $\{\widetilde{\epsilon}\}$ is summarized in Table A.1.

Up to now, we have only investigated the permutation of two neighbouring gradings ϵ_j and ϵ_{j+1}. However, because the result holds for any j, we can also deal with the general situation. Any particular grading pattern with fixed numbers of positive and negative gradings, r and s respectively, can be transformed into any other pattern by repeated nearest-neighbour permutations. Thus, once all Bethe ansatz roots and hole solutions are known for one particular case, Table A.1 can be used to calculate the corresponding roots and holes for any other possible grading choice.

Appendix B

Derivation of the TBA equations for the $sl(n|1)$-symmetric case

In this appendix, we deal with the derivation of the NLIEs for the auxiliary functions $y_1^{(a)}(x)$, which has been omitted in Section 3.4.3.

For $a = 1, \dots, n-1$ and $m = 2, \dots, \infty$ the relations (3.16) also apply here. Therefore, the derivation of the TBA equations for $m \geq 2$ is completely analogous to the $sl(n)$-symmetric case and yields the same result, see (3.37).

The case $m = 1$ is a bit more complicated. Applying the transformation (3.18) to both sides of equation (3.16) for $a = 1, \dots, n-1$ and $m = 1$, which is also valid for the $sl(n|1)$-symmetric case, we eventually obtain

$$\begin{aligned}\widehat{y}_1^{(a)}(k) = \sum_{b=1}^{n-1} \left\{ \widehat{A}_{[n]}^{(a,b)}(k) \widehat{Y}_2^{(b)}(k) - \left(\widehat{A}_{[n]}^{(a,b-1)}(k) + \widehat{A}_{[n]}^{(a,b+1)}(k) \right) \widehat{Y}_1^{(b)}(k) \right\} \\ + \widehat{A}_{[n]}^{(a,1)}(k) \widehat{\phi}_h(k) - \widehat{A}_{[n]}^{(a,n-1)}(k) \left(\widehat{Y}_1^{(n)}(k) - \widehat{y}_1^{(n)}(k) \right) .\end{aligned} \tag{B.1}$$

Additionally, we consider the transform of the function $y_1^{(n)}(x)$,

$$\widehat{y}_1^{(n)}(k) = \widehat{\Lambda}_0^{(n)}(k) - \widehat{\Lambda}_1^{(n-1)}(k) . \tag{B.2}$$

where we still have to eliminate the function $\widehat{\Lambda}_1^{(n-1)}(k)$. In order to achieve this, consider the equation

$$\widehat{\Lambda}_1^{(n-1)}(k) = \widehat{A}_{[n]}^{(n-1,1)}(k) \widehat{\Lambda}_1^{(0)}(k) + \widehat{A}_{[n]}^{(n-1,n-1)}(k) \widehat{\Lambda}_1^{(n)}(k) + \sum_{b=1}^{n-1} \widehat{A}_{[n]}^{(n-1,b)}(k) \widehat{Y}_1^{(b)}(k) , \tag{B.3}$$

which is obtained in analogy to (3.29), and the equation

$$\widehat{Y}_1^{(n)}(k) = \mathrm{e}^{|k|/2} \widehat{\Lambda}_1^{(n)}(k) - \widehat{\Lambda}_1^{(n-1)}(k) , \tag{B.4}$$

which stems from the transform of the function $Y_1^{(n)}(x)$, where the identity

$$\widehat{\Lambda}_1^{(n+1)}(k) = \mathrm{e}^{-|k|/2}\widehat{\Lambda}_1^{(n)}(k) \tag{B.5}$$

has been applied. Note that the latter identity is a direct consequence of the fusion relation

$$\widetilde{\Lambda}_1^{(n+1)}(x-\mathrm{i}/2)\widetilde{\Lambda}_1^{(n+1)}(x+\mathrm{i}/2) = \widetilde{\Lambda}_1^{(n)}(x)\widetilde{\Lambda}_1^{(n+2)}(x)\,. \tag{B.6}$$

Combining equations (B.3) and (B.4) yields

$$\widehat{\Lambda}_1^{(n-1)}(k) = \mathrm{e}^{-(n-1)|k|/2}\widehat{\Lambda}_1^{(0)}(k) + \mathrm{e}^{-(n-1)|k|/2}\sum_{b=1}^{n}\frac{\sinh(bk/2)}{\sinh(k/2)}\widehat{Y}_1^{(b)}(k) - \widehat{Y}_1^{(n)}(k)\,, \tag{B.7a}$$

$$\widehat{\Lambda}_1^{(n)}(k) = \mathrm{e}^{-n|k|/2}\widehat{\Lambda}_1^{(0)}(k) + \mathrm{e}^{-n|k|/2}\sum_{b=1}^{n}\frac{\sinh(bk/2)}{\sinh(k/2)}\widehat{Y}_1^{(b)}(k)\,. \tag{B.7b}$$

We substitute this result into equation (B.2) and obtain

$$\widehat{y}_1^{(n)}(k) = \mathrm{e}^{-n|k|/2}\widehat{\phi}_h(k) - \sum_{b=1}^{n}\mathrm{e}^{-(n-1)|k|/2}\frac{\sinh(bk/2)}{\sinh(k/2)}\widehat{Y}_1^{(b)}(k) + \widehat{Y}_1^{(n)}(k)\,. \tag{B.8}$$

Then, we eliminate $\widehat{y}_1^{(n)}(k)$ from equation (B.1), which leads to

$$\begin{aligned}\widehat{y}_1^{(a)}(k) = {}& \mathrm{e}^{-a|k|/2}\widehat{\phi}_h(k) + \sum_{b=1}^{n-1}\widehat{A}_{[n]}^{(a,b)}(k)\widehat{Y}_2^{(b)}(k) \\ & - \sum_{b=1}^{n}\left(\widehat{A}_{[n]}^{(a,b-1)}(k) + \widehat{A}_{[n]}^{(a,b+1)}(k) + \mathrm{e}^{-(n-1)|k|/2}\frac{\sinh(ak/2)\sinh(bk/2)}{\sinh(k/2)\sinh(nk/2)}\right)\widehat{Y}_1^{(b)}(k)\,.\end{aligned} \tag{B.9}$$

Finally, the set of NLIEs (3.36) results from the inverse transform of equations (B.8) and (B.9).

Appendix C

Derivation of the NLIEs for the $sl(4)$-symmetric case

In this appendix, we give the detailed derivation of the NLIEs for the $sl(4)$-symmetric case that has been deferred from Section 4.2.1.

We apply the transform (4.4) to all auxiliary functions. For brevity, we will treat only the case $k < 0$ here, since the derivation is completely analogous for $k > 0$. For the functions $b^{(a)}_{1,j}(x)$ we get the results

$$\widehat{b}^{(1)}_{1,1}(k) = \mathrm{e}^{k/2}\widehat{\phi}_-(k) - \mathrm{e}^{-k/2}\widehat{X}^{(1)}_2(k)\,, \tag{C.1a}$$

$$\widehat{b}^{(1)}_{1,2}(k) = \mathrm{e}^{k/2}\widehat{\phi}_-(k) + \widehat{X}^{(3)}_2(k) - \mathrm{e}^{-k/2}\widehat{q}^{(h)}_3(k) - \widehat{\Lambda}^{(2)}_1(k)\,, \tag{C.1b}$$

$$\widehat{b}^{(1)}_{1,3}(k) = \mathrm{e}^{k/2}\widehat{\phi}_-(k) + \mathrm{e}^{3k/2}\widehat{q}_2(k) - \mathrm{e}^{k/2}\widehat{q}_3(k) - \widehat{X}^{(2)}_2(k)\,, \tag{C.1c}$$

$$\widehat{b}^{(1)}_{1,4}(k) = \mathrm{e}^{k/2}\widehat{\phi}_-(k) + \mathrm{e}^{3k/2}\widehat{q}_3(k) - \mathrm{e}^{k/2}\widehat{\phi}_+(k) - \mathrm{e}^{k/2}\widehat{X}^{(1)}_1(k)\,, \tag{C.1d}$$

$$\widehat{b}^{(2)}_{1,1}(k) = \mathrm{e}^{k}\widehat{\phi}_-(k) - \mathrm{e}^{-k/2}\widehat{X}^{(2)}_2(k)\,, \tag{C.1e}$$

$$\widehat{b}^{(2)}_{1,2}(k) = \mathrm{e}^{k}\widehat{\phi}_-(k) + \mathrm{e}^{k}\widehat{q}_2(k) - \widehat{X}^{(1)}_1(k) - \mathrm{e}^{-k/2}\widehat{X}^{(3)}_2(k)\,, \tag{C.1f}$$

$$\widehat{b}^{(2)}_{1,3}(k) = \mathrm{e}^{k}\widehat{\phi}_-(k) + \mathrm{e}^{k}\widehat{q}_3(k) - \widehat{q}^{(h)}_2(k) - \widehat{\Lambda}^{(1)}_1(k)\,, \tag{C.1g}$$

$$\widehat{b}^{(2)}_{1,4}(k) = \mathrm{e}^{k}\widehat{\phi}_-(k) + \mathrm{e}^{2k}\widehat{q}_1(k) - \widehat{q}^{(h)}_2(k) - \widehat{\Lambda}^{(3)}_1(k)\,, \tag{C.1h}$$

$$\widehat{b}^{(2)}_{1,5}(k) = \mathrm{e}^{k}\widehat{\phi}_-(k) + \mathrm{e}^{2k}\widehat{q}_1(k) + \mathrm{e}^{k}\widehat{q}_3(k) - \mathrm{e}^{k}\widehat{q}_2(k) - \widehat{X}^{(1)}_2(k) - \mathrm{e}^{k/2}\widehat{X}^{(3)}_1(k)\,, \tag{C.1i}$$

$$\widehat{b}^{(2)}_{1,6}(k) = \mathrm{e}^{k}\widehat{\phi}_-(k) + \mathrm{e}^{2k}\widehat{q}_2(k) - \mathrm{e}^{k/2}\widehat{X}^{(2)}_1(k)\,, \tag{C.1j}$$

$$\widehat{b}^{(3)}_{1,1}(k) = \mathrm{e}^{3k/2}\widehat{\phi}_-(k) - \widehat{X}^{(3)}_1(k)\,, \tag{C.1k}$$

$$\widehat{b}^{(3)}_{1,2}(k) = \mathrm{e}^{3k/2}\widehat{\phi}_-(k) + \mathrm{e}^{k/2}\widehat{q}_3(k) - \widehat{X}^{(2)}_1(k)\,, \tag{C.1l}$$

$$\widehat{b}^{(3)}_{1,3}(k) = \mathrm{e}^{3k/2}\widehat{\phi}_-(k) + \mathrm{e}^{3k/2}\widehat{q}_2(k) + \mathrm{e}^{-k/2}\widehat{X}^{(1)}_2(k) - \mathrm{e}^{3k/2}\widehat{q}_1(k) - \mathrm{e}^{-k/2}\widehat{q}^{(h)}_3(k) - \widehat{\Lambda}^{(2)}_1(k)\,, \tag{C.1m}$$

$$\widehat{b}^{(3)}_{1,4}(k) = \mathrm{e}^{3k/2}\widehat{\phi}_-(k) + \mathrm{e}^{5k/2}\widehat{q}_1(k) - \mathrm{e}^{5k/2}\widehat{\phi}_-(k) - \widehat{X}^{(3)}_2(k)\,. \tag{C.1n}$$

For the uppercase functions $B^{(a)}_{1,j}(x)$ the transform yields

$$\widehat{B}^{(1)}_{1,1}(k) = \mathrm{e}^{-k/2}\widehat{\Lambda}^{(1)}_1(k) - \mathrm{e}^{-k/2}\widehat{X}^{(1)}_2(k)\,, \tag{C.2a}$$
$$\widehat{B}^{(1)}_{1,2}(k) = \mathrm{e}^{-k/2}\widehat{X}^{(1)}_2(k) + \widehat{X}^{(2)}_2(k) - \mathrm{e}^{-k/2}\widehat{q}^{(h)}_3(k) - \widehat{\Lambda}^{(2)}_1(k)\,, \tag{C.2b}$$
$$\widehat{B}^{(1)}_{1,3}(k) = \mathrm{e}^{-k/2}\widehat{q}^{(h)}_3(k) + \mathrm{e}^{k/2}\widehat{X}^{(1)}_1(k) - \mathrm{e}^{k/2}\widehat{q}_3(k) - \widehat{X}^{(2)}_2(k)\,, \tag{C.2c}$$
$$\widehat{B}^{(1)}_{1,4}(k) = \mathrm{e}^{k/2}\widehat{q}_3(k) + \mathrm{e}^{k/2}\widehat{\Lambda}^{(1)}_1(k) - \mathrm{e}^{k/2}\widehat{\phi}_+(k) - \mathrm{e}^{k/2}\widehat{X}^{(1)}_1(k)\,, \tag{C.2d}$$
$$\widehat{B}^{(2)}_{1,1}(k) = \mathrm{e}^{-k/2}\widehat{\Lambda}^{(2)}_1(k) - \mathrm{e}^{-k/2}\widehat{X}^{(2)}_2(k)\,, \tag{C.2e}$$
$$\widehat{B}^{(2)}_{1,2}(k) = \widehat{q}^{(h)}_2(k) + \mathrm{e}^{-k/2}\widehat{X}^{(2)}_2(k) - \widehat{X}^{(1)}_1(k) - \mathrm{e}^{-k/2}\widehat{X}^{(3)}_2(k)\,, \tag{C.2f}$$
$$\widehat{B}^{(2)}_{1,3}(k) = \widehat{X}^{(1)}_1(k) + \widehat{X}^{(1)}_2(k) - \widehat{q}^{(h)}_2(k) - \widehat{\Lambda}^{(1)}_1(k)\,, \tag{C.2g}$$
$$\widehat{B}^{(2)}_{1,4}(k) = \mathrm{e}^{k/2}\widehat{X}^{(3)}_1(k) + \mathrm{e}^{-k/2}\widehat{X}^{(3)}_2(k) - \widehat{q}^{(h)}_2(k) - \widehat{\Lambda}^{(3)}_1(k)\,, \tag{C.2h}$$
$$\widehat{B}^{(2)}_{1,5}(k) = \widehat{q}^{(h)}_2(k) + \mathrm{e}^{k/2}\widehat{X}^{(2)}_1(k) - \mathrm{e}^{k}\widehat{q}_2(k) - \widehat{X}^{(1)}_2(k) - \mathrm{e}^{k/2}\widehat{X}^{(3)}_1(k)\,, \tag{C.2i}$$
$$\widehat{B}^{(2)}_{1,6}(k) = \mathrm{e}^{k}\widehat{q}_2(k) + \mathrm{e}^{k/2}\widehat{\Lambda}^{(2)}_1(k) - \mathrm{e}^{k/2}\widehat{X}^{(2)}_1(k)\,, \tag{C.2j}$$
$$\widehat{B}^{(3)}_{1,1}(k) = \mathrm{e}^{-k/2}\widehat{\Lambda}^{(3)}_1(k) - \widehat{X}^{(3)}_1(k)\,, \tag{C.2k}$$
$$\widehat{B}^{(3)}_{1,2}(k) = \mathrm{e}^{-k/2}\widehat{q}^{(h)}_3(k) + \widehat{X}^{(3)}_1(k) - \widehat{X}^{(2)}_1(k)\,, \tag{C.2l}$$
$$\widehat{B}^{(3)}_{1,3}(k) = \widehat{X}^{(2)}_1(k) + \widehat{X}^{(3)}_2(k) - \mathrm{e}^{3k/2}\widehat{q}_1(k) - \mathrm{e}^{-k/2}\widehat{q}^{(h)}_3(k) - \widehat{\Lambda}^{(2)}_1(k)\,, \tag{C.2m}$$
$$\widehat{B}^{(3)}_{1,4}(k) = \mathrm{e}^{3k/2}\widehat{q}_1(k) + \mathrm{e}^{k/2}\widehat{\Lambda}^{(3)}_1(k) - \mathrm{e}^{5k/2}\widehat{\phi}_-(k) - \widehat{X}^{(3)}_2(k)\,. \tag{C.2n}$$

The latter forms a system of 14 linear equations, where there are exactly 14 unknown functions on the right hand sides. After solving the linear system for the unknown functions, we substitute the result into equations (C.1). This way, only the auxiliary functions survive in the remaining set of equations. After repeating the procedure for the case $k > 0$, we combine the results to get a system of equations valid for all $k \in \mathbb{R}$. The resulting equations are of the form

$$\widehat{b}^{(a)}_{1,j}(k) = -\mathrm{i}N\sinh(kJ\beta/N)\widehat{V}^{(a)}_{[4]}(k) + \sum_{b=1}^{3}\sum_{l=1}^{d_b}\widehat{\mathbf{K}}^{(a,b)}_{j,l}(k)\cdot\widehat{B}^{(b)}_l(k)\,. \tag{C.3}$$

The functions $\widehat{V}^{(a)}_{[n]}(k)$ are given by

$$\widehat{V}^{(a)}_{[n]}(k) = \frac{\sinh([n-a]k/2)}{\sinh(nk/2)}\,, \tag{C.4}$$

and for the matrices $\widehat{\mathbf{K}}^{(a,b)}(k)$ we find the structure

$$\widehat{\mathbf{K}}^{(1,1)}(k) = \begin{pmatrix} \widehat{K}_0(k) & \widehat{K}_1(k) & \widehat{K}_1(k) & \widehat{K}_1(k) \\ \widehat{K}_2(k) & \widehat{K}_0(k) & \widehat{K}_1(k) & \widehat{K}_1(k) \\ \widehat{K}_2(k) & \widehat{K}_2(k) & \widehat{K}_0(k) & \widehat{K}_1(k) \\ \widehat{K}_2(k) & \widehat{K}_2(k) & \widehat{K}_2(k) & \widehat{K}_0(k) \end{pmatrix}\,, \tag{C.5a}$$

$$\widehat{\mathbf{K}}^{(2,2)}(k) = \begin{pmatrix} \widehat{K}_3(k) & \widehat{K}_4(k) & \widehat{K}_4(k) & \widehat{K}_4(k) & \widehat{K}_4(k) & \widehat{K}_6(k) \\ \widehat{K}_5(k) & \widehat{K}_3(k) & \widehat{K}_4(k) & \widehat{K}_4(k) & \widehat{K}_8(k) & \widehat{K}_4(k) \\ \widehat{K}_5(k) & \widehat{K}_5(k) & \widehat{K}_3(k) & \widehat{K}_{10}(k) & \widehat{K}_4(k) & \widehat{K}_4(k) \\ \widehat{K}_5(k) & \widehat{K}_5(k) & \widehat{K}_{10}(k) & \widehat{K}_3(k) & \widehat{K}_4(k) & \widehat{K}_4(k) \\ \widehat{K}_5(k) & \widehat{K}_9(k) & \widehat{K}_5(k) & \widehat{K}_5(k) & \widehat{K}_3(k) & \widehat{K}_4(k) \\ \widehat{K}_7(k) & \widehat{K}_5(k) & \widehat{K}_5(k) & \widehat{K}_5(k) & \widehat{K}_5(k) & \widehat{K}_3(k) \end{pmatrix}, \tag{C.5b}$$

$$\widehat{\mathbf{K}}^{(1,2)}(k) = \begin{pmatrix} \widehat{K}_{11}(k) & \widehat{K}_{11}(k) & \widehat{K}_{11}(k) & \widehat{K}_{12}(k) & \widehat{K}_{12}(k) & \widehat{K}_{12}(k) \\ \widehat{K}_{11}(k) & \widehat{K}_{14}(k) & \widehat{K}_{14}(k) & \widehat{K}_{11}(k) & \widehat{K}_{11}(k) & \widehat{K}_{12}(k) \\ \widehat{K}_{13}(k) & \widehat{K}_{11}(k) & \widehat{K}_{14}(k) & \widehat{K}_{11}(k) & \widehat{K}_{14}(k) & \widehat{K}_{11}(k) \\ \widehat{K}_{13}(k) & \widehat{K}_{13}(k) & \widehat{K}_{11}(k) & \widehat{K}_{13}(k) & \widehat{K}_{11}(k) & \widehat{K}_{11}(k) \end{pmatrix}, \tag{C.5c}$$

$$\widehat{\mathbf{K}}^{(1,3)}(k) = \begin{pmatrix} \widehat{K}_{15}(k) & \widehat{K}_{15}(k) & \widehat{K}_{15}(k) & \widehat{K}_{16}(k) \\ \widehat{K}_{15}(k) & \widehat{K}_{15}(k) & \widehat{K}_{18}(k) & \widehat{K}_{15}(k) \\ \widehat{K}_{15}(k) & \widehat{K}_{19}(k) & \widehat{K}_{15}(k) & \widehat{K}_{15}(k) \\ \widehat{K}_{17}(k) & \widehat{K}_{15}(k) & \widehat{K}_{15}(k) & \widehat{K}_{15}(k) \end{pmatrix}, \tag{C.5d}$$

and

$$\widehat{\mathbf{K}}^{(3,3)}(k) = \widehat{\mathbf{K}}^{(1,1)}(k)\,, \qquad \widehat{\mathbf{K}}^{(2,1)}(k) = \left(\widehat{\mathbf{K}}^{(1,2)}(-k)\right)^{\mathrm{T}}, \tag{C.6a}$$

$$\widehat{\mathbf{K}}^{(3,1)}(k) = \left(\widehat{\mathbf{K}}^{(1,3)}(-k)\right)^{\mathrm{T}}, \qquad \widehat{\mathbf{K}}^{(3,2)}(k) = \left(\widehat{\mathbf{K}}^{(2,3)}(-k)\right)^{\mathrm{T}}, \tag{C.6b}$$

$$\widehat{\mathbf{K}}^{(2,3)}_{j,l}(k) = \widehat{\mathbf{K}}^{(1,2)}_{5-l,7-j}(k)\,. \tag{C.6c}$$

The functions $\widehat{K}_j(k)$ are explicitly given by

$$\widehat{K}_0(k) = \widehat{\mathcal{K}}^{(1,1)}_{[4]}(k)\,, \qquad \widehat{K}_1(k) = \widehat{\mathcal{K}}^{(1,1)}_{[4]}(k) + \mathrm{e}^{-k/2-|k|/2}\,, \tag{C.7a}$$

$$\widehat{K}_2(k) = \widehat{\mathcal{K}}^{(1,1)}_{[4]}(k) + \mathrm{e}^{k/2-|k|/2}\,, \qquad \widehat{K}_3(k) = \widehat{\mathcal{K}}^{(2,2)}_{[4]}(k)\,, \tag{C.7b}$$

$$\widehat{K}_4(k) = \widehat{\mathcal{K}}^{(2,2)}_{[4]}(k) + \mathrm{e}^{-k/2-|k|/2}\,, \qquad \widehat{K}_5(k) = \widehat{\mathcal{K}}^{(2,2)}_{[4]}(k) + \mathrm{e}^{k/2-|k|/2}\,, \tag{C.7c}$$

$$\widehat{K}_6(k) = \widehat{\mathcal{K}}^{(2,2)}_{[4]}(k) + \mathrm{e}^{-k-|k|}\,, \qquad \widehat{K}_7(k) = \widehat{\mathcal{K}}^{(2,2)}_{[4]}(k) + \mathrm{e}^{k-|k|}\,, \tag{C.7d}$$

$$\widehat{K}_8(k) = \widehat{\mathcal{K}}^{(2,2)}_{[4]}(k) + 2\mathrm{e}^{-k/2-|k|/2}\,, \qquad \widehat{K}_9(k) = \widehat{\mathcal{K}}^{(2,2)}_{[4]}(k) + 2\mathrm{e}^{k/2-|k|/2}\,, \tag{C.7e}$$

$$\widehat{K}_{10}(k) = \widehat{\mathcal{K}}^{(2,2)}_{[4]}(k) + \mathrm{e}^{-|k|}\,, \qquad \widehat{K}_{11}(k) = \widehat{\mathcal{K}}^{(1,2)}_{[4]}(k)\,, \tag{C.7f}$$

$$\widehat{K}_{12}(k) = \widehat{\mathcal{K}}^{(1,2)}_{[4]}(k) + \mathrm{e}^{-k-|k|/2} - \mathrm{e}^{-k/2}\,, \qquad \widehat{K}_{13}(k) = \widehat{\mathcal{K}}^{(1,2)}_{[4]}(k) + \mathrm{e}^{k-|k|/2} - \mathrm{e}^{k/2}\,, \tag{C.7g}$$

$$\widehat{K}_{14}(k) = \widehat{\mathcal{K}}^{(1,2)}_{[4]}(k) + \mathrm{e}^{-|k|/2}\,, \qquad \widehat{K}_{15}(k) = \widehat{\mathcal{K}}^{(1,3)}_{[4]}(k)\,, \tag{C.7h}$$

$$\widehat{K}_{16}(k) = \widehat{\mathcal{K}}^{(1,3)}_{[4]}(k) + \mathrm{e}^{-3k/2-|k|/2} - \mathrm{e}^{-k}\,, \qquad \widehat{K}_{17}(k) = \widehat{\mathcal{K}}^{(1,3)}_{[4]}(k) + \mathrm{e}^{3k/2-|k|/2} - \mathrm{e}^{k}\,, \tag{C.7i}$$

$$\widehat{K}_{18}(k) = \widehat{\mathcal{K}}^{(1,3)}_{[4]}(k) + \mathrm{e}^{-k/2-|k|/2} - 1\,, \qquad \widehat{K}_{19}(k) = \widehat{\mathcal{K}}^{(1,3)}_{[4]}(k) + \mathrm{e}^{k/2-|k|/2}\,, \tag{C.7j}$$

with the common function

$$\widehat{\mathcal{K}}^{(a,b)}_{[n]}(k) = \mathrm{e}^{|k|/2}\frac{\sinh(\min(a,b)k/2)\sinh([n-\max(a,b)]k/2)}{\sinh(k/2)\sinh(nk/2)} - \delta_{a,b}\,. \tag{C.8}$$

Since only the first term of (C.3) explicitly depends on the Trotter number N and has the same structure as in the $sl(2)$-symmetric case, we may again use the simple replacement

$$N \sinh(kJ\beta/N) \to kJ\beta \tag{C.9}$$

in order to apply the Trotter limit $N \to \infty$, see equation (4.7). Next we transform the equations back and apply the integration with respect to the spectral parameter x. Doing this, products are replaced by convolutions and we arrive at the final form of the NLIEs given in equations (4.55)–(4.63).

Appendix D

Algebra-related properties of the kernel matrix

In Section 5.1, we have seen that most of the kernel functions of the NLIEs for the general $sl(n)$-symmetric case of the Uimin-Sutherland model can be obtained from reasonable assumptions on the general structure of the equations and by exploiting certain limiting cases of the NLIEs. However, exactly one kernel function of each NLIE cannot be fixed by these considerations. In this appendix, we will therefore explore some additional structure that helps in guessing the remaining functions.

Let us first turn to the constants appearing in the driving terms of the NLIEs for the $sl(n)$-symmetric case of the Uimin-Sutherland model. From equation (5.5) it follows that the constants of the first subset $c_j^{(1)}$ and the first constants of the subsets $c_1^{(a)}$ are explicitly given by

$$c_j^{(1)} = \frac{1}{n}\sum_{k=1}^{n} \mu_k - \mu_j \,, \qquad\qquad c_1^{(a)} = \frac{a}{n}\sum_{k=1}^{n} \mu_k - \sum_{l=1}^{a} \mu_l \,. \tag{D.1}$$

We also introduce the differences α_j between the constants $c_j^{(1)}$ and $c_{j+1}^{(1)}$ which are given by

$$\alpha_j = c_j^{(1)} - c_{j+1}^{(1)} = \mu_{j+1} - \mu_j \,. \tag{D.2}$$

These differences do not only appear in the first subset, but α_j is generally found to be the difference between two constants $c_k^{(a)}$ and $c_l^{(a)}$ for which the corresponding Young tableaux differ only by one entry being j and $j+1$, respectively. We may therefore regard the constants $c_j^{(a)}$ as being weights belonging to vectors of the ath fundamental representation of $sl(n)$ and the differences α_j as being the simple roots[1]. Moreover, there is a connection between the roots α_j, the highest weights $c_1^{(a)}$ for each representation and the Cartan matrix of $sl(n)$,

[1]Thanks to J. Suzuki for pointing this out to me.

$C_{j,k} = -\delta_{j,k-1} - \delta_{j,k+1} + 2\delta_{j,k}$, given by

$$\alpha_j = \sum_{k=1}^{n-1} C_{j,k} \cdot c_1^{(k)} \,. \tag{D.3}$$

It looks tempting to include a spectral parameter dependence and search for a similar structure taking the complete expressions for $\ln b_{1,j}^{(a)}(x)$ as generalized weights. Let us also introduce the generalized roots

$$\alpha_j(x) = \ln b_{1,j}^{(1)}(x) - \ln b_{1,j+1}^{(1)}(x) \,. \tag{D.4}$$

If the expressions $\alpha_j(x)$ are supposed to be proper roots, they should connect the weights $\ln b_{1,j}^{(a)}(x)$ of the other representations as well. Checking this with the explicit form of the auxiliary functions of the $sl(3)$-symmetric case (4.17), however, proves that this can not strictly be the case, since for example

$$\boxed{1} \to \boxed{2}: \quad \alpha_1(x) = \ln b_{1,1}^{(1)}(x) - \ln b_{1,2}^{(1)}(x) \,, \tag{D.5}$$

is clearly different from

$$\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array} \to \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array}: \quad \alpha_1'(x) = \ln b_{1,2}^{(2)}(x) - \ln b_{1,3}^{(2)}(x) \,. \tag{D.6}$$

Note also that equation (D.3) is no longer valid with these definitions. Nevertheless, we observe the functions $\alpha_1(x)$ and $\alpha_1'(x)$ to look very similar to each other if we express them with the help of the final NLIEs (4.23). We find

$$\begin{aligned} \alpha_1(x) = -\beta\alpha_1 &- \left[\left(\frac{\mathrm{i}}{x-\mathrm{i}} - \frac{\mathrm{i}}{x+0\mathrm{i}}\right) * \ln B_{1,1}^{(1)}\right](x) - \left[\left(\frac{\mathrm{i}}{x+\mathrm{i}} - \frac{\mathrm{i}}{x-0\mathrm{i}}\right) * \ln B_{1,2}^{(1)}\right](x) \\ &- \left[\left(\frac{\mathrm{i}}{x-\frac{\mathrm{i}}{2}} - \frac{\mathrm{i}}{x+\frac{\mathrm{i}}{2}}\right) * \ln B_{1,2}^{(2)}\right](x) - \left[\left(\frac{\mathrm{i}}{x+\frac{3}{2}\mathrm{i}} - \frac{\mathrm{i}}{x+\frac{\mathrm{i}}{2}}\right) * \ln B_{1,3}^{(2)}\right](x) \,, \end{aligned} \tag{D.7a}$$

$$\begin{aligned} \alpha_1'(x) = -\beta\alpha_1 &- \left[\left(\frac{\mathrm{i}}{x-\frac{3}{2}\mathrm{i}} - \frac{\mathrm{i}}{x-\frac{\mathrm{i}}{2}}\right) * \ln B_{1,1}^{(1)}\right](x) - \left[\left(\frac{\mathrm{i}}{x+\frac{\mathrm{i}}{2}} - \frac{\mathrm{i}}{x-\frac{\mathrm{i}}{2}}\right) * \ln B_{1,2}^{(1)}\right](x) \\ &- \left[\left(\frac{\mathrm{i}}{x-\mathrm{i}} - \frac{\mathrm{i}}{x+0\mathrm{i}}\right) * \ln B_{1,2}^{(2)}\right](x) - \left[\left(\frac{\mathrm{i}}{x+\mathrm{i}} - \frac{\mathrm{i}}{x-0\mathrm{i}}\right) * \ln B_{1,3}^{(2)}\right](x) \,. \end{aligned} \tag{D.7b}$$

It is a remarkable fact that we can basically recover $\alpha_1'(x)$ just by shifting the argument of all kernel functions in $\alpha_1(x)$ by $-\mathrm{i}/2$. Note that the value for the shift in the argument is not totally unexpected, but that it corresponds to the position of the boxes containing the numbers 1 and 2 in the Young tableaux of the second fundamental representation, confer equation (D.6). Problems occur, however, where the shift leads to a pole at the origin, since $\alpha_1(x)$ contains no information whether the pole should be placed slightly above or below the real axis in the expression for $\alpha_1'(x)$.

The observation that the function $\alpha_1(x)$ is at least similar to a root is not accidental. In analogy, the function $\alpha_2(x)$ is related to $\alpha_2'(x) = \ln b_{1,1}^{(2)} - \ln b_{1,2}^{(2)}$ by a shift of $+\mathrm{i}/2$. We have checked that this observation also holds for the $sl(4)$-symmetric case. Here, the generalized roots $\alpha_j(x)$ are connected to those of the other representations by shifts of $\pm\mathrm{i}/2$ and $\pm\mathrm{i}$ or 0 for the second and third fundamental representations, respectively. Note that the exact positions of the poles near the real axis also need to be neglected here. The structure even holds for the $sl(5)$-symmetric case, as far as the kernel functions can be fixed by exploiting the $sl(4)$ limit. Admitting this in general thus enables us to fix the missing kernel functions from the antidiagonal of the kernel matrix. For the missing kernel functions of the $sl(5)$-symmetric case there is even no ambiguity, since all poles that appear in these functions are away from the origin. A further extension to the $sl(6)$-symmetric case would nevertheless require some additional tests to find the right pole configuration.

Unfortunately, the mathematical background of the observed structure is yet unknown. It is our hope that further investigations may lead to a rigorous construction of the NLIEs for the general $sl(n)$-symmetric case.

Appendix E

Explicit kernel matrix of the $sl(5)$-symmetric case

In this appendix, we give the explicit kernel matrix of the $sl(5)$-symmetric case of the Uimin-Sutherland model that has been deferred from Section 5.1. The submatrices $K_{j,k}^{(a,b)}(x)$ are found to be

$$\mathbf{K}^{(1,1)}(x) = \begin{pmatrix} K_0(x) & K_1(x) & K_1(x) & K_1(x) & K_1(x) \\ K_2(x) & K_0(x) & K_1(x) & K_1(x) & K_1(x) \\ K_2(x) & K_2(x) & K_0(x) & K_1(x) & K_1(x) \\ K_2(x) & K_2(x) & K_2(x) & K_0(x) & K_1(x) \\ K_2(x) & K_2(x) & K_2(x) & K_2(x) & K_0(x) \end{pmatrix}, \tag{E.1a}$$

$$\mathbf{K}^{(1,2)}(x) = \begin{pmatrix} K_3(x) & K_3(x) & K_3(x) & K_3(x) & K_4(x) & K_4(x) & K_4(x) & K_4(x) & K_4(x) & K_4(x) \\ K_3(x) & K_6(x) & K_6(x) & K_6(x) & K_3(x) & K_3(x) & K_3(x) & K_4(x) & K_4(x) & K_4(x) \\ K_5(x) & K_3(x) & K_6(x) & K_6(x) & K_3(x) & K_6(x) & K_6(x) & K_3(x) & K_3(x) & K_4(x) \\ K_5(x) & K_5(x) & K_3(x) & K_6(x) & K_5(x) & K_3(x) & K_6(x) & K_3(x) & K_6(x) & K_3(x) \\ K_5(x) & K_5(x) & K_5(x) & K_3(x) & K_5(x) & K_5(x) & K_3(x) & K_5(x) & K_3(x) & K_3(x) \end{pmatrix}, \tag{E.1b}$$

$$\mathbf{K}^{(1,3)}(x) = \begin{pmatrix} K_7(x) & K_7(x) & K_7(x) & K_7(x) & K_7(x) & K_7(x) & K_8(x) & K_8(x) & K_8(x) & K_8(x) \\ K_7(x) & K_7(x) & K_7(x) & K_{10}(x) & K_{10}(x) & K_{10}(x) & K_7(x) & K_7(x) & K_7(x) & K_8(x) \\ K_7(x) & K_{11}(x) & K_{11}(x) & K_7(x) & K_7(x) & K_{10}(x) & K_7(x) & K_7(x) & K_{10}(x) & K_7(x) \\ K_9(x) & K_7(x) & K_{11}(x) & K_7(x) & K_{11}(x) & K_7(x) & K_7(x) & K_{11}(x) & K_7(x) & K_7(x) \\ K_9(x) & K_9(x) & K_7(x) & K_9(x) & K_7(x) & K_7(x) & K_9(x) & K_7(x) & K_7(x) & K_7(x) \end{pmatrix}, \tag{E.1c}$$

$$\mathbf{K}^{(1,4)}(x) = \begin{pmatrix} K_{12}(x) & K_{12}(x) & K_{12}(x) & K_{12}(x) & K_{13}(x) \\ K_{12}(x) & K_{12}(x) & K_{12}(x) & K_{15}(x) & K_{12}(x) \\ K_{12}(x) & K_{12}(x) & K_{17}(x) & K_{12}(x) & K_{12}(x) \\ K_{12}(x) & K_{16}(x) & K_{12}(x) & K_{12}(x) & K_{12}(x) \\ K_{14}(x) & K_{12}(x) & K_{12}(x) & K_{12}(x) & K_{12}(x) \end{pmatrix}, \tag{E.1d}$$

$$\mathbf{K}^{(2,2)}(x) = \begin{pmatrix} K_{18}(x) & K_{19}(x) & K_{19}(x) & K_{19}(x) & K_{19}(x) & K_{19}(x) & K_{19}(x) & K_{21}(x) & K_{21}(x) & K_{21}(x) \\ K_{20}(x) & K_{18}(x) & K_{19}(x) & K_{19}(x) & K_{19}(x) & K_{23}(x) & K_{23}(x) & K_{19}(x) & K_{19}(x) & K_{21}(x) \\ K_{20}(x) & K_{20}(x) & K_{18}(x) & K_{19}(x) & K_{25}(x) & K_{19}(x) & K_{23}(x) & K_{19}(x) & K_{23}(x) & K_{19}(x) \\ K_{20}(x) & K_{20}(x) & K_{20}(x) & K_{18}(x) & K_{25}(x) & K_{25}(x) & K_{19}(x) & K_{25}(x) & K_{19}(x) & K_{19}(x) \\ K_{20}(x) & K_{20}(x) & K_{25}(x) & K_{25}(x) & K_{18}(x) & K_{19}(x) & K_{19}(x) & K_{19}(x) & K_{19}(x) & K_{21}(x) \\ K_{20}(x) & K_{24}(x) & K_{20}(x) & K_{25}(x) & K_{20}(x) & K_{18}(x) & K_{19}(x) & K_{19}(x) & K_{23}(x) & K_{19}(x) \\ K_{20}(x) & K_{24}(x) & K_{24}(x) & K_{20}(x) & K_{20}(x) & K_{20}(x) & K_{18}(x) & K_{25}(x) & K_{19}(x) & K_{19}(x) \\ K_{22}(x) & K_{20}(x) & K_{20}(x) & K_{25}(x) & K_{20}(x) & K_{20}(x) & K_{25}(x) & K_{18}(x) & K_{19}(x) & K_{19}(x) \\ K_{22}(x) & K_{20}(x) & K_{24}(x) & K_{20}(x) & K_{20}(x) & K_{24}(x) & K_{20}(x) & K_{20}(x) & K_{18}(x) & K_{19}(x) \\ K_{22}(x) & K_{22}(x) & K_{20}(x) & K_{20}(x) & K_{22}(x) & K_{20}(x) & K_{20}(x) & K_{20}(x) & K_{20}(x) & K_{18}(x) \end{pmatrix}, \tag{E.1e}$$

$$\mathbf{K}^{(2,3)}(x) = \begin{pmatrix} K_{26}(x) & K_{26}(x) & K_{26}(x) & K_{27}(x) & K_{27}(x) & K_{27}(x) & K_{27}(x) & K_{27}(x) & K_{27}(x) & K_{30}(x) \\ K_{26}(x) & K_{29}(x) & K_{29}(x) & K_{26}(x) & K_{26}(x) & K_{27}(x) & K_{27}(x) & K_{27}(x) & K_{32}(x) & K_{27}(x) \\ K_{28}(x) & K_{26}(x) & K_{29}(x) & K_{26}(x) & K_{29}(x) & K_{26}(x) & K_{27}(x) & K_{34}(x) & K_{27}(x) & K_{27}(x) \\ K_{28}(x) & K_{28}(x) & K_{26}(x) & K_{28}(x) & K_{26}(x) & K_{26}(x) & K_{36}(x) & K_{27}(x) & K_{27}(x) & K_{27}(x) \\ K_{26}(x) & K_{29}(x) & K_{29}(x) & K_{29}(x) & K_{29}(x) & K_{34}(x) & K_{26}(x) & K_{26}(x) & K_{27}(x) & K_{27}(x) \\ K_{28}(x) & K_{26}(x) & K_{29}(x) & K_{29}(x) & K_{37}(x) & K_{29}(x) & K_{26}(x) & K_{29}(x) & K_{26}(x) & K_{27}(x) \\ K_{28}(x) & K_{28}(x) & K_{26}(x) & K_{35}(x) & K_{29}(x) & K_{29}(x) & K_{28}(x) & K_{26}(x) & K_{26}(x) & K_{27}(x) \\ K_{28}(x) & K_{28}(x) & K_{35}(x) & K_{26}(x) & K_{29}(x) & K_{29}(x) & K_{26}(x) & K_{29}(x) & K_{29}(x) & K_{27}(x) \\ K_{28}(x) & K_{33}(x) & K_{28}(x) & K_{28}(x) & K_{26}(x) & K_{29}(x) & K_{28}(x) & K_{26}(x) & K_{29}(x) & K_{26}(x) \\ K_{31}(x) & K_{28}(x) & K_{28}(x) & K_{28}(x) & K_{28}(x) & K_{26}(x) & K_{28}(x) & K_{28}(x) & K_{26}(x) & K_{26}(x) \end{pmatrix}. \tag{E.1f}$$

Due to the symmetry properties of $\mathbf{K}(x)$, the rest of the submatrices are given by

$$\mathbf{K}^{(2,1)}(x) = \left(\mathbf{K}^{(1,2)}(x)\right)^{\dagger}, \qquad \mathbf{K}^{(2,4)}_{j,k}(x) = \mathbf{K}^{(1,3)}_{6-k,11-j}(x), \tag{E.2a}$$

$$\mathbf{K}^{(3,1)}(x) = \left(\mathbf{K}^{(1,3)}(x)\right)^{\dagger}, \qquad \mathbf{K}^{(3,2)}(x) = \left(\mathbf{K}^{(2,3)}(x)\right)^{\dagger}, \tag{E.2b}$$

$$\mathbf{K}^{(3,3)}_{j,k}(x) = \mathbf{K}^{(2,2)}_{11-k,11-j}(x), \qquad \mathbf{K}^{(3,4)}_{j,k}(x) = \mathbf{K}^{(1,2)}_{6-k,11-j}(x), \tag{E.2c}$$

$$\mathbf{K}^{(4,1)}(x) = \left(\mathbf{K}^{(1,4)}(x)\right)^{\dagger}, \qquad \mathbf{K}^{(4,2)}(x) = \left(\mathbf{K}^{(2,4)}(x)\right)^{\dagger}, \tag{E.2d}$$

$$\mathbf{K}^{(4,3)}(x) = \left(\mathbf{K}^{(3,4)}(x)\right)^{\dagger}, \qquad \mathbf{K}^{(4,4)}_{j,k}(x) = \mathbf{K}^{(1,1)}_{6-k,6-j}(x). \tag{E.2e}$$

The explicit kernel functions are

$$K_0(x) = \mathcal{K}^{(1,1)}_{[5]}(x), \qquad K_1(x) = \mathcal{K}^{(1,1)}_{[5]}(x) + \frac{\mathrm{i}}{x+\mathrm{i}} - \frac{\mathrm{i}}{x-0\mathrm{i}}, \tag{E.3a}$$

$$K_2(x) = \mathcal{K}^{(1,1)}_{[5]}(x) + \frac{\mathrm{i}}{x+0\mathrm{i}} - \frac{\mathrm{i}}{x-\mathrm{i}}, \qquad K_3(x) = \mathcal{K}^{(1,2)}_{[5]}(x), \tag{E.3b}$$

$$K_4(x) = \mathcal{K}^{(1,2)}_{[5]}(x) + \frac{\mathrm{i}}{x+\frac{3}{2}\mathrm{i}} - \frac{\mathrm{i}}{x+\frac{\mathrm{i}}{2}}\,, \qquad K_5(x) = \mathcal{K}^{(1,2)}_{[5]}(x) + \frac{\mathrm{i}}{x-\frac{\mathrm{i}}{2}} - \frac{\mathrm{i}}{x-\frac{3}{2}\mathrm{i}}\,, \tag{E.3c}$$

$$K_6(x) = \mathcal{K}^{(1,2)}_{[5]}(x) + \frac{\mathrm{i}}{x+\frac{\mathrm{i}}{2}} - \frac{\mathrm{i}}{x-\frac{\mathrm{i}}{2}}\,, \qquad K_7(x) = \mathcal{K}^{(1,3)}_{[5]}(x)\,, \tag{E.3d}$$

$$K_8(x) = \mathcal{K}^{(1,3)}_{[5]}(x) + \frac{\mathrm{i}}{x+2\mathrm{i}} - \frac{\mathrm{i}}{x+\mathrm{i}}\,, \qquad K_9(x) = \mathcal{K}^{(1,3)}_{[5]}(x) + \frac{\mathrm{i}}{x-\mathrm{i}} - \frac{\mathrm{i}}{x-2\mathrm{i}}\,, \tag{E.3e}$$

$$K_{10}(x) = \mathcal{K}^{(1,3)}_{[5]}(x) + \frac{\mathrm{i}}{x+\mathrm{i}} - \frac{\mathrm{i}}{x+0\mathrm{i}}\,, \qquad K_{11}(x) = \mathcal{K}^{(1,3)}_{[5]}(x) + \frac{\mathrm{i}}{x+0\mathrm{i}} - \frac{\mathrm{i}}{x-\mathrm{i}}\,, \tag{E.3f}$$

$$K_{12}(x) = \mathcal{K}^{(1,4)}_{[5]}(x)\,, \qquad K_{13}(x) = \mathcal{K}^{(1,4)}_{[5]}(x) + \frac{\mathrm{i}}{x+\frac{5}{2}\mathrm{i}} - \frac{\mathrm{i}}{x+\frac{3}{2}\mathrm{i}}\,, \tag{E.3g}$$

$$K_{14}(x) = \mathcal{K}^{(1,4)}_{[5]}(x) + \frac{\mathrm{i}}{x-\frac{3}{2}\mathrm{i}} - \frac{\mathrm{i}}{x-\frac{5}{2}\mathrm{i}}\,, \qquad K_{15}(x) = \mathcal{K}^{(1,4)}_{[5]}(x) + \frac{\mathrm{i}}{x+\frac{3}{2}\mathrm{i}} - \frac{\mathrm{i}}{x+\frac{\mathrm{i}}{2}}\,, \tag{E.3h}$$

$$K_{16}(x) = \mathcal{K}^{(1,4)}_{[5]}(x) + \frac{\mathrm{i}}{x-\frac{\mathrm{i}}{2}} - \frac{\mathrm{i}}{x-\frac{3}{2}\mathrm{i}}\,, \qquad K_{17}(x) = \mathcal{K}^{(1,4)}_{[5]}(x) + \frac{\mathrm{i}}{x+\frac{\mathrm{i}}{2}} - \frac{\mathrm{i}}{x-\frac{\mathrm{i}}{2}}\,, \tag{E.3i}$$

$$K_{18}(x) = \mathcal{K}^{(2,2)}_{[5]}(x)\,, \qquad K_{19}(x) = \mathcal{K}^{(2,2)}_{[5]}(x) + \frac{\mathrm{i}}{x+\mathrm{i}} - \frac{\mathrm{i}}{x-0\mathrm{i}}\,, \tag{E.3j}$$

$$K_{20}(x) = \mathcal{K}^{(2,2)}_{[5]}(x) + \frac{\mathrm{i}}{x+0\mathrm{i}} - \frac{\mathrm{i}}{x-\mathrm{i}}\,, \qquad K_{21}(x) = \mathcal{K}^{(2,2)}_{[5]}(x) + \frac{\mathrm{i}}{x+2\mathrm{i}} - \frac{\mathrm{i}}{x-0\mathrm{i}}\,, \tag{E.3k}$$

$$K_{22}(x) = \mathcal{K}^{(2,2)}_{[5]}(x) + \frac{\mathrm{i}}{x+0\mathrm{i}} - \frac{\mathrm{i}}{x-2\mathrm{i}}\,, \qquad K_{23}(x) = \mathcal{K}^{(2,2)}_{[5]}(x) + \frac{2\mathrm{i}}{x+\mathrm{i}} - \frac{2\mathrm{i}}{x-0\mathrm{i}}\,, \tag{E.3l}$$

$$K_{24}(x) = \mathcal{K}^{(2,2)}_{[5]}(x) + \frac{2\mathrm{i}}{x+0\mathrm{i}} - \frac{2\mathrm{i}}{x-\mathrm{i}}\,, \qquad K_{25}(x) = \mathcal{K}^{(2,2)}_{[5]}(x) + \frac{\mathrm{i}}{x+\mathrm{i}} - \frac{\mathrm{i}}{x-\mathrm{i}}\,, \tag{E.3m}$$

$$K_{26}(x) = \mathcal{K}^{(2,3)}_{[5]}(x)\,, \qquad K_{27}(x) = \mathcal{K}^{(2,3)}_{[5]}(x) + \frac{\mathrm{i}}{x+\frac{3}{2}\mathrm{i}} - \frac{\mathrm{i}}{x+\frac{\mathrm{i}}{2}}\,, \tag{E.3n}$$

$$K_{28}(x) = \mathcal{K}^{(2,3)}_{[5]}(x) + \frac{\mathrm{i}}{x-\frac{\mathrm{i}}{2}} - \frac{\mathrm{i}}{x-\frac{3}{2}\mathrm{i}}\,, \qquad K_{29}(x) = \mathcal{K}^{(2,3)}_{[5]}(x) + \frac{\mathrm{i}}{x+\frac{\mathrm{i}}{2}} - \frac{\mathrm{i}}{x-\frac{\mathrm{i}}{2}}\,, \tag{E.3o}$$

$$K_{30}(x) = \mathcal{K}^{(2,3)}_{[5]}(x) + \frac{\mathrm{i}}{x+\frac{5}{2}\mathrm{i}} - \frac{\mathrm{i}}{x+\frac{\mathrm{i}}{2}}\,, \qquad K_{31}(x) = \mathcal{K}^{(2,3)}_{[5]}(x) + \frac{\mathrm{i}}{x-\frac{\mathrm{i}}{2}} - \frac{\mathrm{i}}{x-\frac{5}{2}\mathrm{i}}\,, \tag{E.3p}$$

$$K_{32}(x) = \mathcal{K}^{(2,3)}_{[5]}(x) + \frac{2\mathrm{i}}{x+\frac{3}{2}\mathrm{i}} - \frac{2\mathrm{i}}{x+\frac{\mathrm{i}}{2}}\,, \qquad K_{33}(x) = \mathcal{K}^{(2,3)}_{[5]}(x) + \frac{2\mathrm{i}}{x-\frac{\mathrm{i}}{2}} - \frac{2\mathrm{i}}{x-\frac{3}{2}\mathrm{i}}\,, \tag{E.3q}$$

$$K_{34}(x) = \mathcal{K}^{(2,3)}_{[5]}(x) + \frac{\mathrm{i}}{x+\frac{3}{2}\mathrm{i}} - \frac{\mathrm{i}}{x-\frac{\mathrm{i}}{2}}\,, \qquad K_{35}(x) = \mathcal{K}^{(2,3)}_{[5]}(x) + \frac{\mathrm{i}}{x+\frac{\mathrm{i}}{2}} - \frac{\mathrm{i}}{x-\frac{3}{2}\mathrm{i}}\,, \tag{E.3r}$$

$$K_{36}(x) = \mathcal{K}^{(2,3)}_{[5]}(x) + \frac{\mathrm{i}}{x+\frac{3}{2}\mathrm{i}} + \frac{\mathrm{i}}{x-\frac{\mathrm{i}}{2}} - \frac{\mathrm{i}}{x+\frac{\mathrm{i}}{2}} - \frac{\mathrm{i}}{x-\frac{3}{2}\mathrm{i}}\,, \qquad K_{37}(x) = \mathcal{K}^{(2,3)}_{[5]}(x) + \frac{2\mathrm{i}}{x+\frac{\mathrm{i}}{2}} - \frac{2\mathrm{i}}{x-\frac{\mathrm{i}}{2}}\,, \tag{E.3s}$$

where the common function $\mathcal{K}^{(a,b)}_{[5]}(x)$ is the one given in equation (4.62).

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