Nonlinear integral equations for the thermodynamics of the $sl(4)$-symmetric Uimin-Sutherland model

Jens Damerau and Andreas Klümper

1 Abstract

We derive a finite set of nonlinear integral equations (NLIE) for the thermodynamics of the $sl(4)$-symmetric Uimin-Sutherland (US) model. Our NLIE can be numerically evaluated for arbitrary finite temperature and chemical potentials. In contrast to the NLIE of type [2], which have already been generalised to $U_q(\widehat{sl}(m|n))$, the evaluation at small temperatures poses no problem in our formulation. The known nonlinear integral equations for the $sl(3)$ case [1] are recovered as a limiting case. We give numerical results for a spin-orbital model.

2 One-dimensional Uimin-Sutherland model

The Hamiltonian of the one-dimensional US model is given by

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{ext}} = \sum_{j=1}^{L} \pi_{j,j+1} - \sum_{j=1}^{L}\sum_{\alpha=1}^{q} \mu_\alpha n_{j,\alpha}\,. \tag{1}$$

It acts on a one-dimensional lattice with L sites, where a q-state spin variable α_j is assigned to each site j. For each state α we have a grading $\varepsilon_\alpha = (-1)^{p(\alpha)} = \pm 1$. The local interaction operator $\pi_{j,j+1}$ permutes neighbouring spins on the lattice with respect to their grading,

$$\pi_{j,j+1}|\alpha_1 \ldots \alpha_j \alpha_{j+1} \ldots \alpha_L\rangle = (-1)^{p(\alpha_j)p(\alpha_{j+1})}|\alpha_1 \ldots \alpha_{j+1}\alpha_j \ldots \alpha_L\rangle\,, \tag{2}$$

where periodic boundary conditions are imposed. We have added external field terms $\mathcal{H}_{\text{ext}}$, where $n_{j,\alpha}$ counts the number of particles of type α sitting on site j, and μ_α is some general chemical potential. The model is known to be exactly solvable on the basis of the Yang-Baxter algebra. The classical counterpart is the rational limit of the two-dimensional Perk-Schultz model with Boltzmann weights

$$R^{\beta\nu}_{\alpha\mu}(v) = \delta_{\alpha\nu}\delta_{\mu\beta} + v\cdot(-1)^{p(\alpha)p(\mu)}\cdot\delta_{\alpha\beta}\delta_{\mu\nu}\,. \tag{3}$$

We introduce the quantum transfer matrix (QTM),

$$\left(\mathcal{T}^{\text{QTM}}\right)^{\beta}_{\alpha}(v) = \sum_{\{\nu\}} e^{\beta\mu_{\nu_1}} \prod_{j=1}^{N/2} R^{\beta_{2j-1}\nu_{2j}}_{\alpha_{2j-1}\nu_{2j-1}}(iv+u)\widetilde{R}^{\beta_{2j}\nu_{2j+1}}_{\alpha_{2j}\nu_{2j}}(iv-u)\,, \tag{4}$$

where N is the Trotter number, $u = -\beta/N$ and $\widetilde{R}^{\beta\nu}_{\alpha\mu}(v) = R^{\nu\alpha}_{\mu\beta}(-v)$. The partition function of the US model can then be written in terms of the QTM,

$$Z = \operatorname{Tr} e^{-\beta\mathcal{H}} = \lim_{N\to\infty} \operatorname{Tr}\left(\mathcal{T}^{\text{QTM}}(0)\right)^{L}. \tag{5}$$

This implies, that in the thermodynamic limit ($L\to\infty$) the free energy per site is solely given by the unique largest eigenvalue of the QTM at $v=0$ and $N\to\infty$,

$$f = -\lim_{L\to\infty}\frac{1}{L\beta}\ln Z = -\frac{1}{\beta}\ln\Lambda_{\max}(0)\,. \tag{6}$$

The QTM can be diagonalised by use of Bethe ansatz (BA). This yields

$$\Lambda(v) = \sum_{j=1}^{q}\lambda_j(v) = \sum_{j=1}^{q}\phi_-(v)\phi_+(v)\frac{q_{j-1}(v-i\varepsilon_j)}{q_{j-1}(v)}\frac{q_j(v+i\varepsilon_j)}{q_j(v)}e^{\beta\mu_j}\,, \tag{7}$$

where we have defined $\phi_\pm(v) = (v\pm iu)^{N/2}$, $q_0(v) = \phi_-(v)$, $q_q(v) = \phi_+(v)$ and $q_j(v) = \prod_{k_j=1}^{M_j}(v - v^j_{k_j})$ for $j = 1,\ldots,q-1$. For each set j the $M_j \le N/2$ many complex BA roots $v^j_{k_j}$ have to fulfill the BA equations

$$\lambda_j(v^j_{k_j})/\lambda_{j+1}(v^j_{k_j}) = -1\,. \tag{8}$$

The actual numbers M_j depend on the eigenvalue of interest. In the following we are only interested in the largest eigenvalue, for which all $M_j = N/2$.

3 Nonlinear integral equations for the $sl(4)$ case

In general the US model has $sl(m|n)$ symmetry. However NLIE of type [1] were previously known only for $q = m+n \le 3$. Here we treat the $sl(4)$-symmetric US model, for which we have $q = 4$ and all $\varepsilon_j = +1$. The crucial point for the derivation of the NLIE is the knowledge of suitable auxiliary functions. For convenience, we use an abbreviated notation using Young-Tableaux,

$$\boxed{j} = \lambda_j(v)\,, \qquad \begin{array}{|c|}\hline j\\\hline k\\\hline\end{array} = \lambda_j(v-i/2)\lambda_k(v+i/2)\,, \tag{9a}$$

$$\begin{array}{|c|}\hline j\\\hline k\\\hline l\\\hline\end{array} = \lambda_j(v-i)\lambda_k(v)\lambda_l(v+i)\,. \tag{9b}$$

For the first fundamental representation we define four auxiliary functions:

$$b^{(1)}_1(x) = \left.\frac{\boxed{1}}{\boxed{2}+\boxed{3}+\boxed{4}}\right|_{v=x+i/2}, \quad b^{(1)}_4(x) = \left.\frac{\boxed{4}}{\boxed{1}+\boxed{2}+\boxed{3}}\right|_{v=x-i/2}, \tag{10a}$$

$$b^{(1)}_2(x) = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\cdot\left(\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)}{\left(\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}\right)\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)}\right|_{v=x}, \tag{10b}$$

$$b^{(1)}_3(x) = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\cdot\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}\cdot\left(\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)}\right|_{v=x}. \tag{10c}$$

For the second fundamental representation we have six auxiliary functions:

$$b^{(2)}_1(x) = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}}\right|_{v=x+i/2}, \tag{11a}$$

$$b^{(2)}_2(x) = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\cdot\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}}{\left(\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)\cdot\left(\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)}\right|_{v=x+i/2}, \tag{11b}$$

$$b^{(2)}_3(x) = \left.\frac{\boxed{1}\cdot\boxed{4}}{\left(\boxed{2}+\boxed{3}\right)\cdot\left(\boxed{1}+\boxed{2}+\boxed{3}+\boxed{4}\right)}\right|_{v=x}, \tag{11c}$$

$$b^{(2)}_4(x) = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\cdot\begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}}{\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}\right)\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}\right)}\right|_{v=x}, \tag{11d}$$

$$b^{(2)}_5(x) = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\cdot\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}}{\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}\right)\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right)}\right|_{v=x-i/2}, \tag{11e}$$

$$b^{(2)}_6(x) = \left.\frac{\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}}\right|_{v=x-i/2}, \tag{11f}$$

Finally the four auxiliary functions for the third fundamental representation are:

$$b^{(3)}_1(x) = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}}\right|_{v=x+i/2}, \quad b^{(3)}_4(x) = \left.\frac{\begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}}\right|_{v=x-i/2}, \tag{12a}$$

$$b^{(3)}_2(x) = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\cdot\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}}{\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}\right)}\right|_{v=x}, \tag{12b}$$

$$b^{(3)}_3(x) = \left.\frac{\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}\right)}{\left(\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}\right)\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)}\right|_{v=x}. \tag{12c}$$

In addition to the auxiliary functions above, we define a second set of functions, namely $B^{(n)}_j(x) = b^{(n)}_j(x)+1$. By applying a Fourier transform to the logarithmic derivatives of all auxiliary functions $b^{(n)}_j(x)$, $B^{(n)}_j(x)$ and exploiting their analyticity properties in Fourier space, we eventually find a system of coupled nonlinear integral equations, for which we can take the limit $N\to\infty$ analytically. We get

$$\mathbf{b}(x) = -\beta\boldsymbol{\varepsilon}(x) - \left[\underline{\mathbf{K}} * \mathbf{B}\right](x)\,, \tag{13}$$

where we have defined

$$\mathbf{b}(x) = \left(\ln b^{(1)}_1(x),\ldots,\ln b^{(1)}_4(x),\ln b^{(2)}_1(x),\ldots,\ln b^{(3)}_4(x)\right)^{\mathrm{T}}, \tag{14}$$

$$\mathbf{B}(x) = \left(\ln B^{(1)}_1(x),\ldots,\ln B^{(1)}_4(x),\ln B^{(2)}_1(x),\ldots,\ln B^{(3)}_4(x)\right)^{\mathrm{T}}, \tag{15}$$

$$\boldsymbol{\varepsilon}(x) = \left(\varepsilon^{(1)}_1(x),\ldots,\varepsilon^{(1)}_4(x),\varepsilon^{(2)}_1(x),\ldots,\varepsilon^{(3)}_4(x)\right)^{\mathrm{T}}. \tag{16}$$

Convolutions are denoted by

$$\left[f*g\right](x) = \int_{-\infty}^{\infty} f(x-y)g(y)\frac{dy}{2\pi}. \tag{17}$$

The kernel matrix $\underline{\mathbf{K}}(x)$ is a 14 by 14 matrix. We divide the matrix into blocks connecting the auxiliary functions from different representations,

$$\underline{\mathbf{K}}(x) = \begin{pmatrix} \underline{\mathbf{K}}^{(1,1)}(x) & \underline{\mathbf{K}}^{(1,2)}(x) & \underline{\mathbf{K}}^{(1,3)}(x)\\ \underline{\mathbf{K}}^{(2,1)}(x) & \underline{\mathbf{K}}^{(2,2)}(x) & \underline{\mathbf{K}}^{(2,3)}(x)\\ \underline{\mathbf{K}}^{(3,1)}(x) & \underline{\mathbf{K}}^{(3,2)}(x) & \underline{\mathbf{K}}^{(3,3)}(x)\end{pmatrix}. \tag{18}$$

We observe, that $\underline{\mathbf{K}}(x)$ is hermitian and invariant under reflection along the anti-diagonal ($[\underline{\mathbf{K}}(x)]_{j,k} = [\underline{\mathbf{K}}(x)]_{15-k,15-j}$). Therefore we only need to consider

$$\underline{\mathbf{K}}^{(1,1)}(x) = \begin{pmatrix} K_0(x) & K_1(x) & K_1(x) & K_1(x)\\ K_2(x) & K_0(x) & K_1(x) & K_1(x)\\ K_2(x) & K_2(x) & K_0(x) & K_1(x)\\ K_2(x) & K_2(x) & K_2(x) & K_0(x)\end{pmatrix}, \tag{19a}$$

$$\underline{\mathbf{K}}^{(2,2)}(x) = \begin{pmatrix} K_3(x) & K_4(x) & K_4(x) & K_4(x) & K_4(x) & K_6(x)\\ K_5(x) & K_3(x) & K_4(x) & K_4(x) & K_8(x) & K_4(x)\\ K_5(x) & K_5(x) & K_3(x) & K_{10}(x) & K_4(x) & K_4(x)\\ K_5(x) & K_5(x) & K_{10}(x) & K_3(x) & K_4(x) & K_4(x)\\ K_5(x) & K_9(x) & K_5(x) & K_5(x) & K_3(x) & K_4(x)\\ K_7(x) & K_5(x) & K_5(x) & K_5(x) & K_5(x) & K_3(x)\end{pmatrix}, \tag{19b}$$

$$\underline{\mathbf{K}}^{(1,2)}(x) = \begin{pmatrix} K_{11}(x) & K_{11}(x) & K_{11}(x) & K_{12}(x) & K_{12}(x) & K_{12}(x)\\ K_{11}(x) & K_{14}(x) & K_{14}(x) & K_{11}(x) & K_{11}(x) & K_{12}(x)\\ K_{13}(x) & K_{11}(x) & K_{14}(x) & K_{11}(x) & K_{14}(x) & K_{11}(x)\\ K_{13}(x) & K_{13}(x) & K_{11}(x) & K_{13}(x) & K_{11}(x) & K_{11}(x)\end{pmatrix}, \tag{19c}$$

$$\underline{\mathbf{K}}^{(1,3)}(x) = \begin{pmatrix} K_{15}(x) & K_{15}(x) & K_{15}(x) & K_{16}(x)\\ K_{15}(x) & K_{15}(x) & K_{18}(x) & K_{15}(x)\\ K_{15}(x) & K_{19}(x) & K_{15}(x) & K_{15}(x)\\ K_{17}(x) & K_{15}(x) & K_{15}(x) & K_{15}(x)\end{pmatrix}. \tag{19d}$$

The kernels $K_j(x)$ are defined as $K_j(x) = \int_{-\infty}^{\infty}\widehat{K}_j(k)e^{ikx}\,dk$, where

$$\begin{aligned}
\widehat{K}_0(k) &= \widehat{K}^{(1,1)}_{[4]}(k)\,, & \widehat{K}_1(k) &= \widehat{K}^{(1,1)}_{[4]}(k) + e^{-k/2-|k|/2}\,,\\
\widehat{K}_2(k) &= \widehat{K}^{(1,1)}_{[4]}(k) + e^{k/2-|k|/2}\,, & \widehat{K}_3(k) &= \widehat{K}^{(2,2)}_{[4]}(k)\,,\\
\widehat{K}_4(k) &= \widehat{K}^{(2,2)}_{[4]}(k) + e^{-k/2-|k|/2}\,, & \widehat{K}_5(k) &= \widehat{K}^{(2,2)}_{[4]}(k) + e^{k/2-|k|/2}\,,\\
\widehat{K}_6(k) &= \widehat{K}^{(2,2)}_{[4]}(k) + e^{-k-|k|}\,, & \widehat{K}_7(k) &= \widehat{K}^{(2,2)}_{[4]}(k) + e^{k-|k|}\,,\\
\widehat{K}_8(k) &= \widehat{K}^{(2,2)}_{[4]}(k) + 2e^{-k/2-|k|/2}\,, & \widehat{K}_9(k) &= \widehat{K}^{(2,2)}_{[4]}(k) + 2e^{k/2-|k|/2}\,,\\
\widehat{K}_{10}(k) &= \widehat{K}^{(2,2)}_{[4]}(k) + e^{-|k|}\,, & \widehat{K}_{11}(k) &= \widehat{K}^{(1,2)}_{[4]}(k)\,,\\
\widehat{K}_{12}(k) &= \widehat{K}^{(1,2)}_{[4]}(k) + e^{-k-|k|/2} - e^{-k/2}\,, & \widehat{K}_{13}(k) &= \widehat{K}^{(1,2)}_{[4]}(k) + e^{k-|k|/2} - e^{k/2}\,,\\
\widehat{K}_{14}(k) &= \widehat{K}^{(1,2)}_{[4]}(k) + e^{-|k|/2}\,, & \widehat{K}_{15}(k) &= \widehat{K}^{(1,3)}_{[4]}(k)\,,\\
\widehat{K}_{16}(k) &= \widehat{K}^{(1,3)}_{[4]}(k) + e^{-3k/2-|k|/2} - e^{-k}\,, & \widehat{K}_{17}(k) &= \widehat{K}^{(1,3)}_{[4]}(k) + e^{3k/2-|k|/2} - e^{k}\,,\\
\widehat{K}_{18}(k) &= \widehat{K}^{(1,3)}_{[4]}(k) + e^{-k/2-|k|/2} - 1\,, & \widehat{K}_{19}(k) &= \widehat{K}^{(1,3)}_{[4]}(k) + e^{k/2-|k|/2}\,,
\end{aligned} \tag{20}$$

with the function

$$\widehat{K}^{(i,j)}_{[q]}(k) = \frac{\sum_{m=0}^{i-1}\sum_{n=0}^{q-j-1}e^{-(m+n+(j-i)/2)|k|}}{\sum_{n=0}^{q-1}e^{-n|k|}} - \delta_{ij}\,. \tag{21}$$

We note, that in spectral parameter space all kernels can be written in terms of digamma and simple rational functions. Nevertheless our notation is more useful here, as the numerical treatment of the NLIE can conveniently be done in Fourier space. The bare energies in (16) are $\varepsilon^{(n)}_j(x) = V^{(n)}_{[4]}(x) + c^{(n)}_j$, where

$$V^{(n)}_{[q]}(x) = \frac{2\pi}{q}\frac{\sin(\pi n/q)}{\cosh(2\pi x/q)-\cos(\pi n/q)}\,, \tag{22}$$

and the constants are given by

$$\begin{aligned}
c^{(1)}_1 &= (-3\mu_1+\mu_2+\mu_3+\mu_4)/4\,, & c^{(1)}_2 &= (\mu_1-3\mu_2+\mu_3+\mu_4)/4\,, & &(23a)\\
c^{(1)}_3 &= (\mu_1+\mu_2-3\mu_3+\mu_4)/4\,, & c^{(1)}_4 &= (\mu_1+\mu_2+\mu_3-3\mu_4)/4\,, & &(23b)\\
c^{(2)}_1 &= (-\mu_1-\mu_2+\mu_3+\mu_4)/2\,, & c^{(2)}_2 &= (-\mu_1+\mu_2-\mu_3+\mu_4)/2\,, & &(23c)\\
c^{(2)}_3 &= (-\mu_1+\mu_2+\mu_3-\mu_4)/2\,, & c^{(2)}_4 &= (\mu_1-\mu_2-\mu_3+\mu_4)/2\,, & &(23d)\\
c^{(2)}_5 &= (\mu_1-\mu_2+\mu_3-\mu_4)/2\,, & c^{(2)}_6 &= (\mu_1+\mu_2-\mu_3-\mu_4)/2\,, & &(23e)\\
c^{(3)}_1 &= (-\mu_1-\mu_2-\mu_3+3\mu_4)/4\,, & c^{(3)}_2 &= (-\mu_1-\mu_2+3\mu_3-\mu_4)/4\,, & &(23f)\\
c^{(3)}_3 &= (-\mu_1+3\mu_2-\mu_3-\mu_4)/4\,, & c^{(3)}_4 &= (3\mu_1-\mu_2-\mu_3-\mu_4)/4\,. & &(23g)
\end{aligned}$$

Finally the largest eigenvalue of the QTM can be written in terms of the auxiliary functions,

$$\ln\Lambda_{\max}(0) = -\beta\left(1-\frac{\pi}{4}-\frac{3}{2}\ln 2-\frac{1}{4}\sum_{j=1}^{4}\mu_j\right) + \sum_{n=1}^{3}\sum_{j=1}^{d_n}\left[V^{(n)}_{[4]} * \ln B^{(n)}_j\right](0)\,, \tag{24}$$

where d_n is the dimension of the n-th fundamental representation. Therefore the problem of solving the infinitely many BA equations (8) in the limit $N\to\infty$ has been reduced to finding a finite set of functions satisfying the NLIE (13)–(23). The NLIE is valid for arbitrary finite temperature and chemical potentials.

4 Analytical investigation of the $sl(3)$ limit

We want to show, how our formulation (13)–(24) reduces to the known NLIE for the $sl(3)$-symmetric case by freezing out one of the states. We choose the state $\alpha = 4$ and accordingly treat the limit $\mu_4\to-\infty$. We observe, that only seven of the auxiliary functions survive. We can regard

$$b^{(1)}_4(x) \equiv b^{(2)}_3(x) \equiv b^{(2)}_5(x) \equiv b^{(2)}_6(x) \equiv b^{(3)}_2(x) \equiv b^{(3)}_3(x) \equiv b^{(3)}_4(x) \equiv 0\,. \tag{25}$$

We also conclude, that $b^{(3)}_1(x)/B^{(3)}_1(x)\to 1$. Using this information the equation for $\ln b^{(3)}_1(x)$ linearises and can be solved analytically. Substituting this into our NLIE, we are again left with a NLIE of type (13) but with only six auxiliary functions belonging to the two fundamental representations of $sl(3)$. Here we get the kernel matrix

$$\underline{\mathbf{K}}(x) = \begin{pmatrix} K_0(x) & K_1(x) & K_1(x) & K_3(x) & K_3(x) & K_4(x)\\ K_2(x) & K_0(x) & K_1(x) & K_3(x) & K_6(x) & K_3(x)\\ K_2(x) & K_2(x) & K_0(x) & K_5(x) & K_3(x) & K_3(x)\\ K_3(x) & K_3(x) & K_4(x) & K_0(x) & K_1(x) & K_1(x)\\ K_3(x) & K_6(x) & K_3(x) & K_2(x) & K_0(x) & K_1(x)\\ K_5(x) & K_3(x) & K_3(x) & K_2(x) & K_2(x) & K_0(x)\end{pmatrix}. \tag{26}$$

The kernels are given in Fourier space as

$$\begin{aligned}
\widehat{K}_0(k) &= \widehat{K}^{(1,1)}_{[3]}(k)\,, & \widehat{K}_1(k) &= \widehat{K}^{(1,1)}_{[3]}(k) + e^{-k/2-|k|/2}\,,\\
\widehat{K}_2(k) &= \widehat{K}^{(1,1)}_{[3]}(k) + e^{k/2-|k|/2}\,, & \widehat{K}_3(k) &= \widehat{K}^{(1,2)}_{[3]}(k)\,,\\
\widehat{K}_4(k) &= \widehat{K}^{(1,2)}_{[3]}(k) + e^{-k-|k|/2} - e^{-k/2}\,, & \widehat{K}_5(k) &= \widehat{K}^{(1,2)}_{[3]}(k) + e^{k-|k|/2} - e^{k/2}\,,\\
\widehat{K}_6(k) &= \widehat{K}^{(1,2)}_{[3]}(k) + e^{-|k|/2}\,.
\end{aligned} \tag{27}$$

For the bare energies we get $\varepsilon^{(n)}_j(x) = V^{(n)}_{[3]}(x) + c^{(n)}_j$ with the constants

$$\begin{aligned}
c^{(1)}_1 &= (-2\mu_1+\mu_2+\mu_3)/3\,, & c^{(1)}_2 &= (\mu_1-2\mu_2+\mu_3)/3\,, & &(28a)\\
c^{(1)}_3 &= (\mu_1+\mu_2-2\mu_3)/3\,, & c^{(2)}_1 &= (-\mu_1-\mu_2+2\mu_3)/3\,, & &(28b)\\
c^{(2)}_2 &= (-\mu_1+2\mu_2-\mu_3)/3\,, & c^{(2)}_3 &= (2\mu_1-\mu_2-\mu_3)/3\,. & &(28c)
\end{aligned}$$

The largest eigenvalue is given by

$$\ln\Lambda_{\max}(0) = -\beta\left(1-\frac{\pi}{3\sqrt{3}}-\ln 3-\frac{1}{3}\sum_{j=1}^{3}\mu_j\right) + \sum_{n=1}^{2}\sum_{j=1}^{3}\left[V^{(n)}_{[3]} * \ln B^{(n)}_j\right](0)\,. \tag{29}$$

As expected, this is exactly the known NLIE for the $sl(3)$-symmetric case [1].

5 Numerical results for a spin-orbital model

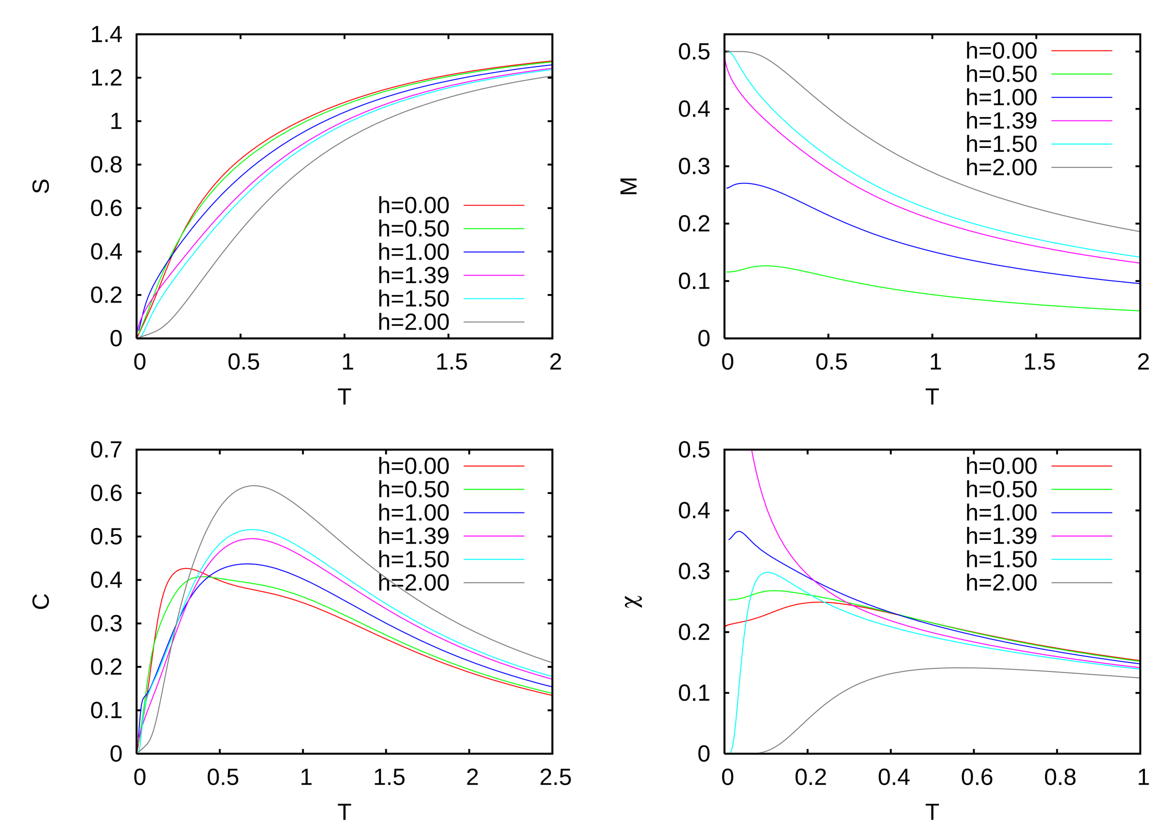
As an application, we consider the Hamiltonian of a $SU(2)\times SU(2)$ spin-orbital model at the supersymmetric point,

$$\mathcal{H} = \sum_{j=1}^{L}(2\boldsymbol{S}_j\boldsymbol{S}_{j+1}+1/2)(2\boldsymbol{\tau}_j\boldsymbol{\tau}_{j+1}+1/2) - \sum_{j=1}^{L}\left(g_S h S^z_j + g_\tau h\tau^z_j\right). \tag{30}$$

We have allowed for an external magnetic field h, which couples to the spins and orbital pseudo-spins with Landé factors g_S and g_τ respectively. Clearly the Hamiltonian is equivalent to the $sl(4)$ US model Hamiltonian (1), if we set

$$\begin{aligned}
\mu_1 &= (g_S+g_\tau)h/2\,, & \mu_2 &= (g_S-g_\tau)h/2\,, & &(31a)\\
\mu_3 &= -(g_S-g_\tau)h/2\,, & \mu_4 &= -(g_S+g_\tau)h/2\,. & &(31b)
\end{aligned}$$

In the four figures below, results are shown for the entropy S, specific heat C, magnetisation M and magnetic susceptibility χ in the case $g_S = 1$, $g_\tau = 0$ for various magnetic fields. There is a critical field, which is known to be $h_c = 2\ln 2 \approx 1.39$. For $h > h_c$ all spins are fully polarised in the ground state.



References

[1] A. Fujii and A. Klümper, *Anti-symmetrically fused model and non-linear integral equations in the three-state Uimin-Sutherland model*, Nucl. Phys. B **546** (1999), 751.

[2] Z. Tsuboi, *Nonlinear integral equations and high temperature expansion for the $U_q(\widehat{sl}(r+1|s+1))$ Perk-Schultz model*, Nucl. Phys. B **737** (2006), 261.