Exact Thermodynamics of the Uimin-Sutherland Model via Nonlinear Integral Equations

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1. Introduction
   - Definition of the Uimin-Sutherland model
   - Applications of the model
2. Derivation of nonlinear integral equations
   - Quantum transfer matrix approach to thermodynamics
   - Definition of suitable auxiliary functions
   - Finite set of nonlinear integral equations
3. Numerical results
   - $SU(4)$ spin-orbital model
   - Essler-Korepin-Schoutens model
4. Summary and outlook

Definition of the Uimin-Sutherland model

Hamiltonian

$$\mathcal{H}_0 = J \sum_{j=1}^{L} \pi_{j,j+1}$$

- One-dimensional lattice with L sites, periodic boundary conditions.
- Each site j carries a spin variable $\alpha_j \in \{1, \dots, q\}$.
- Local Hamiltonian $\pi_{j,j+1}$ is the graded permutation operator,

$$\pi_{j,j+1} |\dots \alpha_j \, \alpha_{j+1} \, \dots\rangle = (-1)^{p(\alpha_j)p(\alpha_{j+1})} |\dots \alpha_{j+1} \, \alpha_j \, \dots\rangle \, .$$

- Invariant under graded symmetry group $SU(r|s)$, where $q = r + s$.

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$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{ext}} = J \sum_{j=1}^{L} \pi_{j,j+1} - \sum_{j=1}^{L} \sum_{\alpha=1}^{q} \mu_\alpha n_{j,\alpha}$$

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- Invariant under graded symmetry group $SU(r|s)$, where $q = r + s$.
- External fields may be added via generic chemical potentials μ_α.

Applications of the model

Many interesting models are of Uimin-Sutherland type.

- Spin-1/2 Heisenberg chain: $[q = 2, r = 2, s = 0]$
  - $\mathcal{H}_0 = J \sum_{j=1}^{L} (2\boldsymbol{S}_j \boldsymbol{S}_{j+1} + 1/2)$

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  - $\mathcal{H}_0 = J\sum_{j=1}^{L}(2\boldsymbol{S}_j\boldsymbol{S}_{j+1}+1/2)$
- Supersymmetric t-J model ($2t=J$): $[q=3, r=2, s=1]$
  - $\mathcal{H}_0 = t\sum_{j=1}^{L}\sum_{\sigma}\mathcal{P}(c_{j,\sigma}^{\dagger}c_{j+1,\sigma}+c_{j,\sigma}c_{j+1,\sigma}^{\dagger})\mathcal{P}+J\sum_{j=1}^{L}(\boldsymbol{S}_j\boldsymbol{S}_{j+1}-n_jn_{j+1}/4)$

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- $SU(4)$ spin-orbital model: $[q = 4, r = 4, s = 0]$
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- Essler-Korepin-Schoutens model: $[q = 4, r = 2, s = 2]$
  - $$\mathcal{H}_0 = J \sum_{j=1}^{L} [(c_{j,\uparrow}^{\dagger} c_{j+1,\uparrow} + c_{j,\uparrow} c_{j+1,\uparrow}^{\dagger})(1 - n_{j,\downarrow} - n_{j+1,\downarrow}) + \ldots$$

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2. Construct a suitable 'quantum' transfer matrix (QTM).
   $\Rightarrow \quad Z = \lim_{N\to\infty} \mathrm{Tr}(\mathcal{T}_{\mathrm{QTM}}(0))^L$

$$(\mathcal{T}_{QTM})^{\{\alpha'\}}_{\{\alpha\ \}}(v) = \sum_{\{\nu\}} \mathrm{e}^{\beta\mu_{\nu_1}} \prod_{j=1}^{N/2} R^{\alpha'_{2j-1}\nu_{2j}}_{\alpha_{2j-1}\nu_{2j-1}}(\mathrm{i}v - \beta/N)\widetilde{R}^{\alpha'_{2j}\nu_{2j+1}}_{\alpha_{2j}\nu_{2j}}(\mathrm{i}v + \beta/N)$$

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3. Consider large systems: Thermodynamic limit.
   $$\Rightarrow \quad f = -\lim_{L\to\infty} \frac{1}{L\beta} \ln Z = -\frac{1}{\beta} \lim_{N\to\infty} \ln \Lambda_{\max}(0)$$

Diagonalization of the quantum transfer matrix

- Quantum transfer matrices form a commuting family:

$$[\mathcal{T}_{\mathsf{QTM}}(v), \mathcal{T}_{\mathsf{QTM}}(v')] = 0 \quad \text{for all } v, v' \in \mathbb{C}\,.$$

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$$\Lambda(v) = \sum_{j=1}^{q} \lambda_j(v) = \sum_{j=1}^{q} \Phi_-(v)\Phi_+(v) \frac{q_{j-1}(v - \mathrm{i}\epsilon_j)}{q_{j-1}(v)} \frac{q_j(v + \mathrm{i}\epsilon_j)}{q_j(v)} \mathrm{e}^{\beta\mu_j}\,.$$

- Bethe ansatz equations ensure that eigenvalues are free of poles:

$$\frac{\lambda_j(v^j_{k_j})}{\lambda_{j+1}(v^j_{k_j})} = -1 \quad \text{for all roots } v^j_{k_j} \text{ of polynomials } q_j(v)\,.$$

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- Largest eigenvalue: $N(q-1)/2$ many coupled nonlinear equations.

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- $[q = 2, r = 2, s = 0]$: 2 auxiliary b-functions
  - $b_1^{(1)}(v) = \dfrac{\lambda_1(v + \mathrm{i}/2)}{\lambda_2(v + \mathrm{i}/2)}, \quad b_2^{(1)}(v) = \dfrac{\lambda_2(v - \mathrm{i}/2)}{\lambda_1(v - \mathrm{i}/2)}$

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- $[q = 3, r = 2, s = 1]$: 3 auxiliary b-functions
  - $b_1^{(1)}(v) = \left.\frac{\lambda_1}{\lambda_2 + \lambda_3}\right|_{(v+\mathrm{i}/2)}$, $b_2^{(1)}(v) = \left.\frac{\lambda_3}{\lambda_1 + \lambda_2}\right|_{(v-\mathrm{i}/2)}$,

    $b_1^{(2)}(v) = \left.\frac{\lambda_1 \lambda_3}{\lambda_2 \left(\lambda_1 + \lambda_2 + \lambda_3\right)}\right|_{(v)}$

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- $[q = 2, r = 2, s = 0]$: 2 auxiliary b-functions
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- $[q = 4, r = 4, s = 0]$: 14 auxiliary b-functions
- $[q = 4, r = 2, s = 2]$: 6 auxiliary b-functions

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1. Zeros and poles of functions $b_j^{(n)}$ and $b_j^{(n)} + 1$ located in groups along slightly curved lines of approximately same imaginary parts.

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3. All functions $\ln b_j^{(n)}$ can be written in terms of $\ln(b_j^{(n)} + 1)$. Closed set of equations, $N \to \infty$ is possible.

Finite set of nonlinear integral equations

Generic form

$$\ln b_j^{(n)}(v) = -\beta(V^{(n)}(v) + c_j^{(n)}) - \sum_{k,m} \Big[K_{j,k}^{(n,m)} * \ln(b_k^{(m)} + 1) \Big](v)$$

$$\ln \Lambda_{\max}(0) = -\beta e_0 + \sum_{j,n} \Big[V^{(n)} * \ln(b_j^{(n)} + 1) \Big](0)$$

- Admits analytical solution in high- and low-temperature limits.
- Numerical solution by iteration using Fast Fourier Transform algorithm to calculate convolutions.

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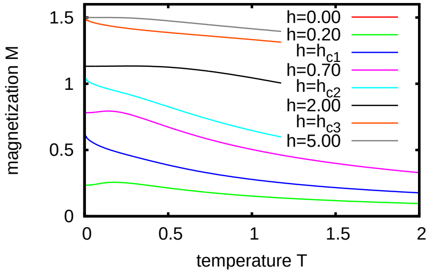
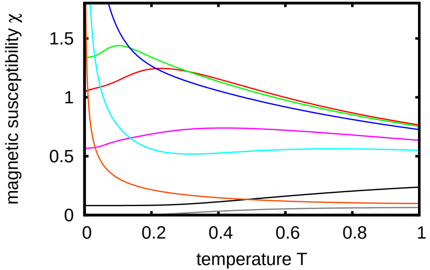
- Numerical solution by iteration using Fast Fourier Transform algorithm to calculate convolutions.
- Good numerical precision: Typically around 4 to 6 digits.
- Derivatives can also be calculated directly:
  - Entropy, specific heat, magnetization, susceptibilities, ...

$SU(4)$ spin-orbital model at $g_S = 2$, $g_\tau = 1$

Numerical results





Magnetic field couples to spins and orbitals:

- $\mathcal{H} = \mathcal{H}_0 - g_S h S^z - g_\tau h \tau^z$, with $g_S = 2$, $g_\tau = 1$
- Basis vectors: $|\uparrow_S \uparrow_\tau\rangle$, $|\uparrow_S \downarrow_\tau\rangle$, $|\downarrow_S \uparrow_\tau\rangle$, $|\downarrow_S \downarrow_\tau\rangle$
- Three critical fields:
  - $h_{c1} \approx 0.370$
  - $h_{c2} \approx 0.941$
  - $h_{c3} = 4$
- Divergence of the magnetic susceptibility at all critical fields

$SU(4)$ spin-orbital model at $g_S = 2$, $g_\tau = 1$

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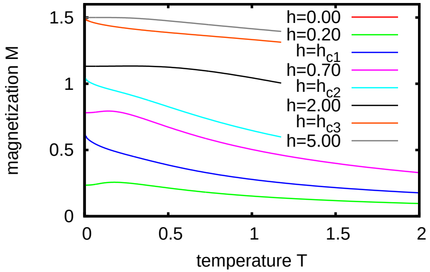

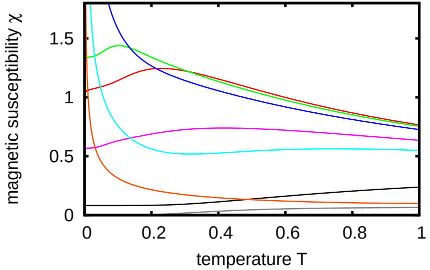

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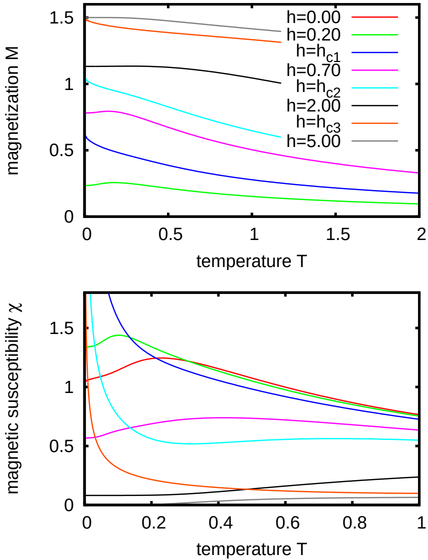

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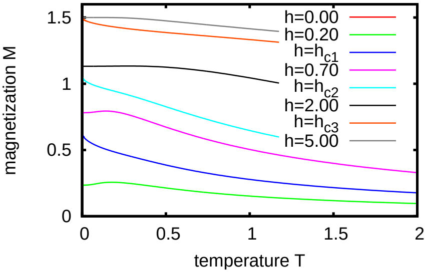

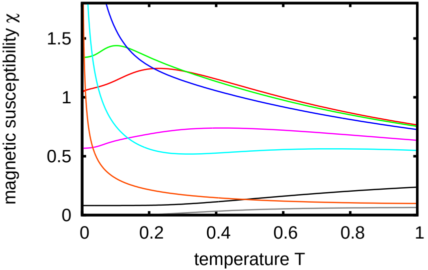

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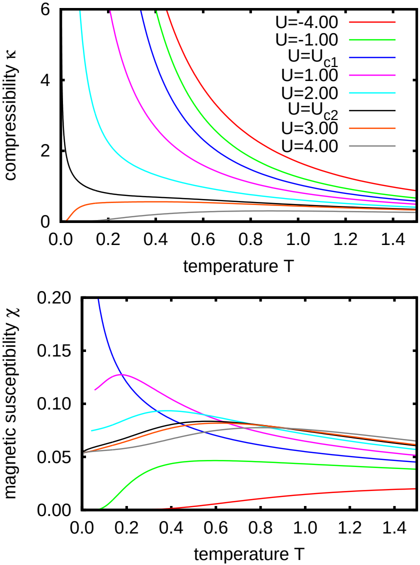
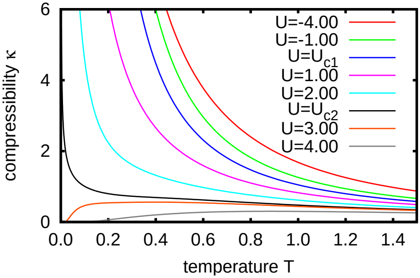
Essler-Korepin-Schoutens model at half filling

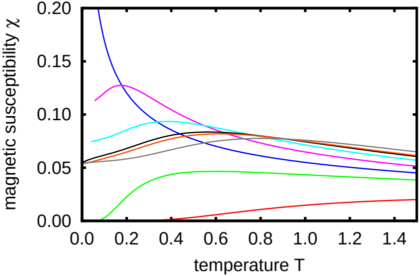
Numerical results



Additional chemical potential and Hubbard parameter:

- $\mathcal{H} = \mathcal{H}_0 - \mu\mathcal{N}$
  $+U\sum_j (n_{j,\uparrow} - 1/2)(n_{j,\downarrow} - 1/2)$
- Basis vectors:
  $|\uparrow\rangle$, $|\downarrow\rangle$, $|\uparrow\downarrow\rangle$, $|0\rangle$
- Fixed density $n(\mu) = 1.0$
- Critical Hubbard parameters:
  - $U_{c1} = 0$
  - $U_{c2} = 4\ln 2 \approx 2.77$





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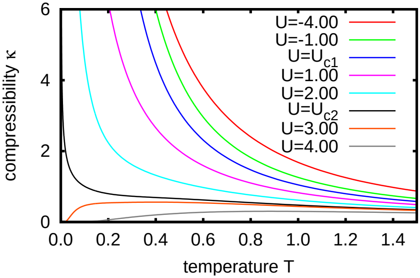

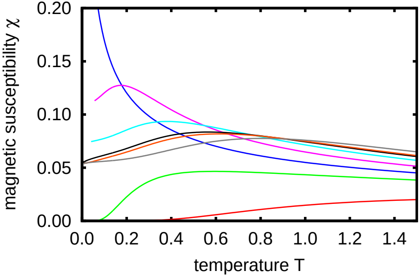

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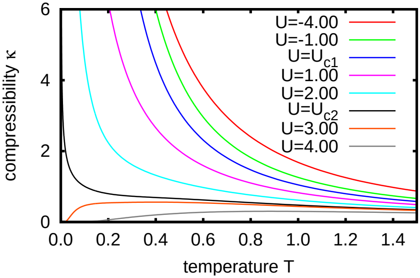

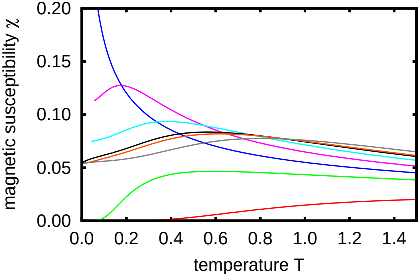

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Summary and outlook

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  - Nonlinear integral equations are well suited for numerical as well as analytical treatment of the Uimin-Sutherland model
  - Known for physically interesting models ($q \leq 4$ & all gradings)
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- Outlook
  - Generalization to arbitrary US models is still difficult (Conjectures for $q = 5$)
  - Current work: $SU(4|1)$ spin-orbital model with mobile defects

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- Outlook
  - Generalization to arbitrary US models is still difficult (Conjectures for $q = 5$)
  - Current work: $SU(4|1)$ spin-orbital model with mobile defects
- Further reading
  - J. Damerau and A. Klümper, J. Stat. Mech. (2006) P12014

$$\begin{aligned}\mathcal{H} = J\sum_{j=1}^{L}[&(c_{j,\uparrow}^{\dagger}c_{j+1,\uparrow} + c_{j,\uparrow}c_{j+1,\uparrow}^{\dagger})(1 - n_{j,\downarrow} - n_{j+1,\downarrow})\\
&+ (c_{j,\downarrow}^{\dagger}c_{j+1,\downarrow} + c_{j,\downarrow}c_{j+1,\downarrow}^{\dagger})(1 - n_{j,\uparrow} - n_{j+1,\uparrow})\\
&+ (n_j - 1)(n_{j+1} - 1)/2 - (n_{j,\uparrow} - n_{j,\downarrow})(n_{j+1,\uparrow} - n_{j+1,\downarrow})/2\\
&+ c_{j,\uparrow}^{\dagger}c_{j,\downarrow}^{\dagger}c_{j+1,\downarrow}c_{j+1,\uparrow} + c_{j,\uparrow}c_{j,\downarrow}c_{j+1,\downarrow}^{\dagger}c_{j+1,\uparrow}^{\dagger}\\
&- c_{j,\downarrow}^{\dagger}c_{j,\uparrow}c_{j+1,\uparrow}^{\dagger}c_{j+1,\downarrow} - c_{j,\uparrow}^{\dagger}c_{j,\downarrow}c_{j+1,\downarrow}^{\dagger}c_{j+1,\uparrow}\\
&+ (n_{j,\uparrow} - 1/2)(n_{j,\downarrow} - 1/2) + (n_{j+1,\uparrow} - 1/2)(n_{j+1,\downarrow} - 1/2)]\\
+ U\sum_{j=1}^{L}&(n_{j,\uparrow} - 1/2)(n_{j,\downarrow} - 1/2) - \mu\mathcal{N}\end{aligned}$$

$$b_1^{(1)}(v) = \left.\frac{\boxed{1}}{\boxed{2}+\boxed{3}+\boxed{4}}\right|_{v+\mathrm{i}/2} \qquad b_4^{(1)}(v) = \left.\frac{\boxed{4}}{\boxed{1}+\boxed{2}+\boxed{3}}\right|_{v-\mathrm{i}/2}$$

$$b_2^{(1)}(v) = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\cdot\left(\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)}{\left(\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}\right)\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)}\right|_{v}$$

$$b_3^{(1)}(v) = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\cdot\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}\cdot\left(\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)}\right|_{v}$$

$$b_1^{(2)}(v) = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}}\right|_{v+\mathrm{i}/2}$$

$$b_2^{(2)}(v) = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\cdot\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}}{\left(\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)\cdot\left(\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)}\right|_{v+\mathrm{i}/2}$$

$$b_3^{(2)}(v) = \left.\frac{\boxed{1}\cdot\boxed{4}}{(\boxed{2}+\boxed{3})\cdot(\boxed{1}+\boxed{2}+\boxed{3}+\boxed{4})}\right|_{v}$$

$$b_4^{(2)}(v) = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\cdot\begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}}{\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}\right)\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}\right)}\right|_{v}$$

$$b_5^{(2)}(v) = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\cdot\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}}{\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}\right)\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right)}\right|_{v-\mathrm{i}/2}$$

$$b_6^{(2)}(v) = \left.\frac{\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}}\right|_{v-\mathrm{i}/2}$$

$$b_1^{(3)}(v) = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}}\right|_{v+\mathrm{i}/2} \qquad b_4^{(3)}(v) = \left.\frac{\begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}}\right|_{v-\mathrm{i}/2}$$

$$b_2^{(3)}(v) = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\cdot\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}}{\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}\right)}\right|_{v}$$

$$b_3^{(3)}(v) = \left.\frac{\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}\right)}{\left(\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}\right)\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)}\right|_{v}$$

$$\boxed{j} = \lambda_j(v) \qquad \begin{array}{|c|}\hline j\\\hline k\\\hline\end{array} = \lambda_j(v-\mathrm{i}/2)\lambda_k(v+\mathrm{i}/2) \qquad \begin{array}{|c|}\hline j\\\hline k\\\hline l\\\hline\end{array} = \lambda_j(v-\mathrm{i})\lambda_k(v)\lambda_l(v+\mathrm{i}) \qquad (j \le k \le l)$$

$$b_1^{(1)}(v) = \left.\frac{\boxed{1}\!\boxed{1} + \boxed{1}\!\boxed{2} + \boxed{1}\!\boxed{3} + \boxed{1}\!\boxed{4} + \boxed{2}\!\boxed{3} + \boxed{2}\!\boxed{4} + \boxed{3}\!\boxed{4} + \boxed{4}\!\boxed{4}}{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 2\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 3\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}}\right|_v$$

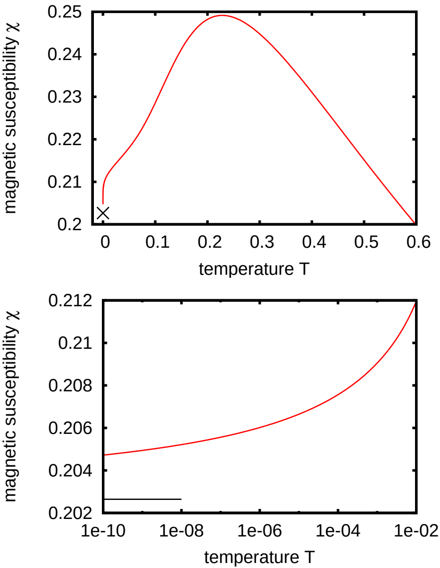
$$b_{2,1}^{(1)}(v) = \left.\frac{\boxed{1}\!\boxed{1} + \boxed{1}\!\boxed{2} + \boxed{1}\!\boxed{3} + \boxed{1}\!\boxed{4}}{\boxed{2}\!\boxed{3} + \boxed{2}\!\boxed{4} + \boxed{3}\!\boxed{4} + \boxed{4}\!\boxed{4}}\right|_{v+\mathrm{i}/2} \qquad b_{2,2}^{(1)}(v) = \left.\frac{\boxed{1}\!\boxed{4} + \boxed{2}\!\boxed{4} + \boxed{3}\!\boxed{4} + \boxed{4}\!\boxed{4}}{\boxed{1}\!\boxed{1} + \boxed{1}\!\boxed{2} + \boxed{1}\!\boxed{3} + \boxed{2}\!\boxed{3}}\right|_{v-\mathrm{i}/2}$$

$$b_{1,1}^{(2)}(v) = \left.\frac{\boxed{1} + \boxed{2} + \boxed{3} + \boxed{4}}{\boxed{1} + \boxed{2}}\right|_v \cdot \left.\frac{\begin{array}{|c|}\hline 3\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}}{\boxed{2}\!\boxed{3} + \boxed{2}\!\boxed{4} + \boxed{3}\!\boxed{4} + \boxed{4}\!\boxed{4}}\right|_{v+\mathrm{i}/2}$$

$$b_{1,2}^{(2)}(v) = \left.\frac{\boxed{1} + \boxed{2} + \boxed{3} + \boxed{4}}{\boxed{3} + \boxed{4}}\right|_v \cdot \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 2\\\hline\end{array}}{\boxed{1}\!\boxed{1} + \boxed{1}\!\boxed{2} + \boxed{1}\!\boxed{3} + \boxed{2}\!\boxed{3}}\right|_{v-\mathrm{i}/2}$$

$$b_2^{(2)}(v) = \left.\frac{\boxed{1}\!\boxed{4} \cdot \left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 2\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 3\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)}{\boxed{2}\!\boxed{3} \cdot \left(\boxed{1}\!\boxed{1} + \boxed{1}\!\boxed{2} + \boxed{1}\!\boxed{3} + \boxed{1}\!\boxed{4} + \boxed{2}\!\boxed{3} + \boxed{2}\!\boxed{4} + \boxed{3}\!\boxed{4} + \boxed{4}\!\boxed{4}\right)}\right|_v$$

$$\boxed{j} = \lambda_j(v) \qquad \begin{array}{|c|}\hline j\\\hline k\\\hline\end{array} = \lambda_j(v - \mathrm{i}/2)\lambda_k(v + \mathrm{i}/2) \qquad \boxed{j}\!\boxed{k} = \lambda_j(v + \mathrm{i}/2)\lambda_k(v - \mathrm{i}/2)$$

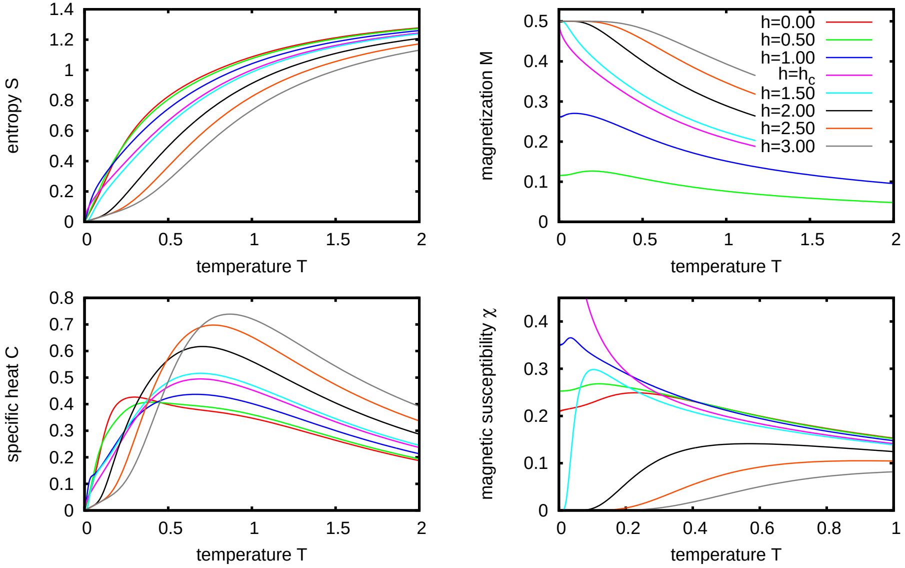


Magnetic susceptibility at $h = 0$:

- Ground-state value known from conformal field theory: $\chi(0) = 2/\pi^2 \approx 0.2026$
- At $T = 10^{-10}$ still well above that value
- Infinite slope due to logarithmic corrections

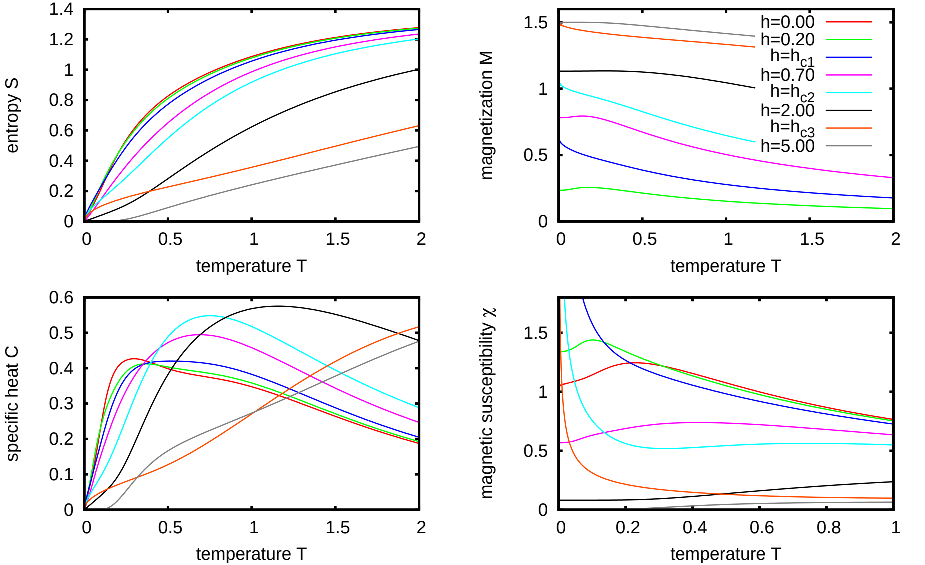
$SU(4)$ spin-orbital model at $g_S = 1$, $g_\tau = 0$

Appendix



$SU(4)$ spin-orbital model at $g_S = 2$, $g_\tau = 1$

Appendix



Essler-Korepin-Schoutens model at half filling

Appendix